Artículo de Investigación. Revista Ciencia e Ingeniería. Vol. 46, No. 2, 143-156, abril-julio, 2025.
ISSN 1316-7081. ISSN Elect. 2244-8780 Universidad de los Andes (ULA)

Qualitative analysis of the predator-prey model two-dimensional (R^2)
Análisis cualitativo del modelo predador-presa Dos-Dimensiones (R^2)

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Abstract

In this work, a generalized global qualitative analysis procedure for planar systems is proposed using the nonlinear dynamic system called Predator-Prey. To obtain information on the global behavior, the behavior of the trajectories around the finite equilibrium points is analyzed by through of Linearization and parametric analysis, then the equilibrium points are located at infinity using ***Poincaré Compactification*** *thechnique to study the behavior of the trajectories around them, and finally, using the development of the Continuous Dependence of the solutions with respect to the Initial Conditions and Parameters, a brief introduction is made about the flow of the trajectories of the system. With these three methodologies, the Poincaré Disks are obtained for each operating range of the system, where the set of disks allows us to demonstrate the overall qualitative behavior of the system. The proposed procedureis based on Qualitative theory, which allows obtaining characteristic information of non-linear systems.*

Keywords: Nonlinear Systems · Equilibrium Points · Dynamic Systems · Poincaré Compactification ·Predator-Prey

Resumen

En este trabajo se plantea un procedimiento generalizado de analisis cualitativo global para sistemas planares cuyo caso de estudio es un sistema dinámico no lineal denominado Presa-Depredador. Para obtener la información del comportamiento global, se analiza el comportamiento de las trayectorias alrededor de los puntos de equilibrio finitos por medio de la Linealización y del análisis paramétrico, luego se localizan los puntos de equilibrio en el infinito por medio de la ***Compactificación de Poincaré*** *para luego estudiar el comportamiento de las trayectorias alrededor de los mismos, y para finalizar, usando el desarrollo de la Dependencia Continua de las soluciones respecto de las Condiciones Iniciales y Parámetros se hace una breve introducción acerca del flujo de las trayectorias del sistema. Con estas tres metodologías se obtienen los Discos de Poincaré para cada rango de operación del sistema, donde el conjunto de los discos permite demostrar cual es el comportamiento cualitativo global del mismo. El procedimiento planteado se basa en la teoría Cualitativa, la cual permite obtener información característica importante de los sistemas no lineales siendo este el aporte de este trabajo.*

Palabras clave: Sistemas no lineales · Puntos de Equilibrios · Sistemas Dinamicos · Compactificación de Poincaré · Predador-Presa.

1 Introduction

In the absence of an explicit analytical solution $x(t)$ of a nonlinear system $\dot{x} = f(t,x)$, one option is the use of the qualitative theory of differential equations. This uses different techniques such as Linearization and the study in the environment of equilibria, normalization through linear transformations, parametric analysis of the systems, Poincaré Compactification, Theorem and Lemmas of hyperbolic, elliptic and parabolic equilibria, among others to achieve classification. Different behaviors of the nonlinear system that are represented through phase portraits, with which it is possible to obtain a representative atlas that classifies the behaviors.

As a case study of a nonlinear system $\dot{x} = f(t,x)$, the Predator-Prey model was chosen due to its abudance in the literature and since it is basic in the study of the operations research area.

It begins with a review of the literature. We continue with the presentation of a technique for global analysis of the behavior of a planar system, in this case the techniques of Linearization and Poincaré Compactification for the study of finite and infinite equilibrium points. The Predator-Prey model is presented, a linear transformation is applied to normalize it, a parametric analysis is carried out with which different behavior of the balances are classified through of thorems, corollaries and remark. An analysis of the trajectories is carried out, culminating in some conjectures for the flow of the nonlinear system and the phase portraits are presented through of the Poincaré disks that make up the atlas.

The main idea is to present an article that serves as an introductory tool for the use of the qualitative theory of dynamic systems.

2 Literature Review

In the 19th century, works were published referring to the formalization of general methods for the theory of nonlinear differential equations, where the work of Poincaré (1881) *Les Méthods nouvelles de la mécanique céleste* and Lyapunov (1892) *General problem of stability of motion* (in Russian), which gave rise to the development of the foundations of what is now *called qualitative theory of non-linear differential equations*.

The development of the qualitative theory of nonlinear systems has been fueled in part by the problems proposed by Hilbert (1902), specifically, the search for how many limit cycles has a system of quadratic differential equations and has served as a guideline in the development of the analysis of said quadratic systems, although it is still an open problem. Along these same lines, Coppel (1966) develops a study providing the phase portraits of some planar quadratic systems and characterizes them in terms of algebraic inequalities between their coefficients, and Dickson and Perko (1970) study planar quadratic systems with bounded trajectories called *Bounded Quadratic Systems (BQS)*.

Likewise, Napoles (2004) in his analysis explains that the qualitative theory is based on the fact that it is possible to visualize the displacement of a point, where from an initial position, and for each value of t and x, the slope of its tangent line at the point (t,x) is given by the derivative expression $\dot{x} = f(t,x)$. These trajectories are called *flow lines* and the shape that these curves take around the equilibrium points gives important information about the behavior of the systems modeled by these equations. The set of these trajectories is called *Phase Portraits or Phase Diagrams*.

On the other hand, the study of population dynamics is a classic area of applied mathematics that dates back to the beginning of the 20th century, and which has given raise to such significant advances as the theory of bifurcations and chaos. The Predator-Prey model has been and is the object of study in population dynamics theory. Cano and Pestana (2011) study models of a single species, linear and nonlinear models of two species, models for n species and non-autonomous systems, that is, explicitly dependent on time; on some of them they apply Lyapunov stability analysis and study their bifurcations.

In this sense, there are six publications that are of special interest in this research: Poincaré (1881), Andronov (1973), Dickson and Perko (1970), Dumortier et al. (2006), Bolaños and Llibre (2014) and Diz-Pita et al. (2022), where all include the *Poincaré Compactification* technique which is a tool that allows studying the behavior of trajectories in the vicinity of infinity. Through it, it is possible to obtain, in some cases, global behaviors of non-linear differential equations through the *Poincaré Disks*.

There is detailed information on the development of this work that was published by Mendez-Díaz (2017) where the procedures condensed in this article are explicitly shown.

3 Global Behavior of a Planar System

The analysis of the *finite equilibrium points* through

Linearization, and the simulations of trajectories through numerical methods are the most used tools to analyze the qualitative behavior of a dynamic system of the form

$$\dot{x} = f(t, x) \quad (1)$$

Both techniques have the disadvantage of being local, the first because the behavior analysis is carried out in a vicinity of the finite equilibrium point and the second because it is a single simulation in a space of infinite trajectories. In view of these disadvantages the question arises, *how does space behave that does not correspond to either finite equilibrium points or simulations?*

In the attempt to answer the previous question, the so-called *Poincaré Compactification* is included as a tool, which allows locating the *equilibrium points at infinity* and analyzing the behavior in their vicinity.

Adding to the above, another question arises, *how do trajectories behave when they come or go from infinity, from or towards the finite equilibrium points?* The Theorems referring to the Continuous Dependence of the solutions with respect to the Initial Conditions and Parameters defined and published by Hartman (1973) provide this information and complement a set of mathematical tools that make it possible to perform a global qualitative analysis of the behavior of the system (1).

It is noted that the procedure proposed in this work is not general for all nonlinear systems; however, it allows addressing some problems, especially if (1) is a planar system, that is, a system of differential equations of second order made up of polynomial functions.

In this research work, use is made of the Linearization method described in Dumortier et al. (2006), section 1.5, pp 14. Also, some theorems will be used as tools that explain the behavior of the equilibrium points of a system through the eigenvalues, which are: the *Hyperbolic Singular Point Theorem* (*Theorem* 11) and the *Semi-Hyperbolic Singular Point Theorem* (*Theorem* 12), this are developed in Dumortier et al (2006) pp 71 and 74 respectively, which explain the behavior of isolated singular points.

In the case that the singular points are not isolated, that is, they form an infinite line of equilibrium points, the Theorem (13) will be used: *Normally Hyperbolic Invariant Manifold Theorem*, developed by Llibre et al. (2013) (*Theorem* 3 pp 234).

1.1 Finite Equilibrium Points

To locate and analyze the behavior of the finite equilibrium points, it is proposed to follow the following steps:

1. To find the equilibrium points of the (1) system, its dynamic part is set to zero, this implies that $\dot{x} = 0$, as a consequence the expression becomes $f(x) = 0$, solving the resulting system of equations gives the set called *finite equilibrium points*, denoted as $\bar{x}$.

2. If the equilibrium points obtained are isolated, the following is done:
   a. Since $f(x) = 0$, we proceed to perform the Linearization of the system, finding the matrix A since by $A(x) = \dfrac{\partial f(x)}{\partial x}$.
   b. Then A is evaluated at the set of equilibrium points $\bar{x}$, and thus a linear system is obtained in a circumviron of each equilibrium point, that is, $A(\bar{x}) = A(x)\big|_{\bar{x}}$.
   c. With $A(x)\big|_{\bar{x}}$ the eigenvalues of the equivalent linear system are obtained for each equilibrium point, called $\lambda_i(\alpha_1, \dots, \alpha_n)$ for $i = 1, \dots, n$ where $\alpha_1, \dots, \alpha_n$ are the parameters of the (1) system.
   d. The *Theorem* 11 and the *Theorem* 12 are used, together with the $\lambda_i(\alpha_1, \dots, \alpha_n)$, to obtain the behaviors of said finite equilibrium points of the linearized system.

3. If the equilibrium points are not isolated, that is, there is a line of infinite points of equilibrium points; the *Theorem* 13 is used.

1.2 Infinite Equilibrium Points

1.2.1 Poincaré Compactification

If the functions that define the vector field X are polynomial, then it is possible to apply *Poincaré Compactification* which shows the trajectories that tend to, or come from, infinity. Let (P, Q) be a polynomial vector field where the functions P and Q are polynomials of arbitrary degree d in the variables x_1 and x_2 of the system,

$$\dot{x}_1 = P(x_1, x_2), \quad \dot{x}_2 = Q(x_1, x_2), \quad (2)$$

where d is the maximum of the degrees of P and Q. To construct the compactification the field X is projects over a unitary radio sphere in both hemispheres or over the equatorial unitary disk. The infinity region projection of field X correspond to radio unitary circumference. This procedure is called *Poincaré Compactification*.

To study the Poincaré Compactification of (P, Q) on the sphere $S^2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ six local charts are represented like: $U_k = \{y \in S^2 : y_k > 0\}$, $V_k = \{y \in S^2 : y_k < 0\}$ for $k = 1,2,3$. The local chart $U_1, U_2, U_3, V_1, V_2, V_3$ are the projection sphere over six planes perpendicular to the sphere (see Fig. 1).

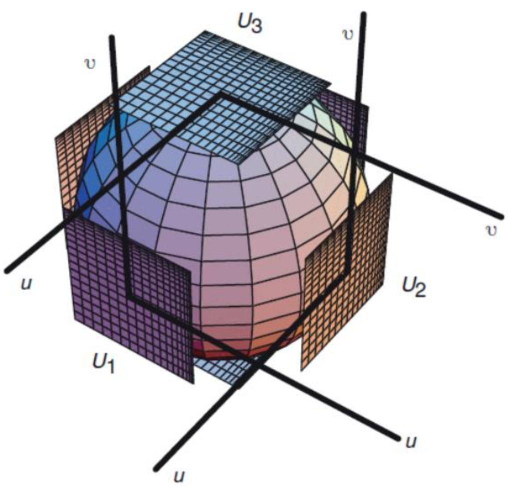

Fig. 1. Local Charts in Poincaré Compactification.

On the local chart U_1 with coordinates (u,v) for $x_1 = 1/v$, $x_2 = u/v$ the expression of the Poincaré Compactification of (P,Q) is

$$\begin{aligned} \dot{u} &= v^d \left[-uP\left(\frac{1}{v},\frac{u}{v}\right) + Q\left(\frac{1}{v},\frac{u}{v}\right) \right], \\ \dot{v} &= -v^{d+1} P\left(\frac{1}{v},\frac{u}{v}\right). \end{aligned} \tag{3}$$

On the local chart U_2 with coordinates (u,v) for $x_1 = u/v$, $x_2 = 1/v$ the expression of the Poincaré Compactification of (P,Q) is

$$\begin{aligned} \dot{u} &= v^d \left[P\left(\frac{u}{v},\frac{1}{v}\right) - uQ\left(\frac{u}{v},\frac{1}{v}\right) \right], \\ \dot{v} &= -v^{d+1} Q\left(\frac{u}{v},\frac{1}{v}\right), \end{aligned} \tag{4}$$

and in the local letter U_3 the expression of the Poincaré Compactification of (P,Q) is

$$\dot{u} = P(u,v), \qquad \dot{v} = Q(u,v). \tag{5}$$

The expression for charts V_k is the same as for U_k multiplied by $(-1)^{d-1}$ for $k = 1,2,3$.

1.2.2 Equilibrium Points at Infinity

According to Dumortier et al. (2006), the singular points at infinity of $\rho(X)$ are the singular points of $\rho(X)$ that are located in S^1. If $y \in S^1$ is an infinite singular point, then $-y$ is also a singular point. Since the local behavior near $-y$ is the local behavior near y multiplied by $(-1)^{d-1}$, it follows that the orientation of the orbits changes when the degree of the vector field is even. For example, if d is even and $y \in S^1$ is a stable node of $\rho(X)$, then $-y$ is an unstable node. Because the equilibrium points at infinity appear in pairs of diametrically opposite points (antipodal points), it is sufficient to study half of them, and with the degree of the vector field the behavior of the other half can be determined.

To study the phase portrait at the equilibrium points of infinity, a point at infinity $(u,0)$ is chosen and the linear part of the field $\rho(X)$ is used. P and Q are denoted by the homogeneous polynomials of degree i for $i = 0,1,\ldots,d$ such that $P = P_0 + P_1 + \ldots + P_d$ and $Q = Q_0 + Q_1 + \ldots + Q_d$. Then $(u,0) \in S^1 \cap (U_1 \cup V_1)$ is an equilibrium point at infinity of $\rho(X)$ if and only if $F(u) \equiv Q_d(1,u) - uP_d(1,u) = 0$.

Likewise $(u,0) \in S^1 \cap (U_2 \cup V_2)$ is an infinity equilibrium point of $\rho(X)$ if and only if $G(u) \equiv P_d(u,1) - uQ_d(u,1) = 0$. The Jacobian matrix of the vector field $\rho(X)$ is also used at the point $(u,0)$ is

$$\begin{bmatrix} F'(u) & Q_{d-1}(1,u) - uP_{d-1}(1,u) \\ 0 & -P_d(1,u) \end{bmatrix}$$

or

$$\begin{bmatrix} G'(u) & P_{d-1}(u,1) - uQ_{d-1}(u,1) \\ 0 & -Q_d(u,1) \end{bmatrix},$$

if $(u,0)$ belongs to $U_1 \cup V_1$ or $U_2 \cup V_2$, respectively. The equator of S^2 can consist entirely of equilibrium points, but in most cases the equilibrium points are isolated. In this case, the study is limited to isolated equilibrium points.

1.2.3 Procedure to Use Poincaré Compactification

To locate and analyze the equilibrium points at infinity, the following steps must be followed:

– The expression for the local letter U_1 is obtained by substituting $x_1 = 1/v$ and $x_2 = u/v$ in the system (2), from which it is obtained

$$P\left(\frac{1}{v},\frac{u}{v}\right),\quad Q\left(\frac{1}{v},\frac{u}{v}\right),$$

this are then substituted into (3) which generates the local card expression U_1. Once the expression for the local letter U_1 is obtained, the Linearization procedure is repeated to find a linear equivalent, this procedure is shown in Section 3.1.

- The expression for the local card U_2 is obtained by substituting $x_1 = u/v$ and $x_2 = 1/v$ into the system (2), from which it is obtain

$$P\left(\frac{u}{v},\frac{1}{v}\right),\quad Q\left(\frac{u}{v},\frac{1}{v}\right),$$

this are then substituted into (4), yielding the local chart expression U_2. Once the expression in the local chart U_2 is obtained, the Linearization procedure is repeated to find a linear equivalent.

- Finally the expression on the local cards V_k are equal to the expression of U_k multiplied by $(-1)^{d-1}$ for $k = 1,2,3$.

4 Case study: Model Predator-Prey $\mathbb{R}^2$

The case of two species, called predator and prey, that coexist in a common ecosystem is considered. The number (or density) of the prey and predator species, respectively, is represented by x_1 and x_2. The model initially proposed by Volterra (2011) is expressed as follows:

$$\begin{aligned}\dot{x}_1 &= x_1(a - bx_2),\\ \dot{x}_2 &= x_2(dx_1 - c),\end{aligned}\tag{6}$$

where a,b,c and d are constants. It will be assumed that the initially defined constants will also take both positive and negative values, since the purpose is to study the model globally. In order to reduce some parameters of the system (6), a matrix Linear Transformation $x = \Omega z$ is performed, defined by

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}\omega_{11} & \omega_{12}\\ \omega_{21} & \omega_{22}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}.\tag{7}$$

Substituting (6) and the matrix $\Omega = \begin{bmatrix}0 & 1/d\\ 1/b & 0\end{bmatrix}$ in the derivative of (7), the system of differential equations in terms of the variable z takes the form

$$\begin{aligned}\dot{z}_1 &= z_1 z_2 - cz_1,\\ \dot{z}_2 &= -z_2 z_1 + az_2.\end{aligned}\tag{8}$$

2.1 *Finite Equilibrium Points*

The *equilibrium points* of the system (8) are represented by $\overline{z}_1$ and $\overline{z}_2$ and are defined as

$$p_1(\overline{z}_1,\overline{z}_2) = (0,0),\quad p_2(\overline{z}_1,\overline{z}_2) = (a,c),\tag{9}$$

obtaining that the equilibrium points are isolated and the Linearization method is applied.

The Jacobian matrix of the system (8) is $A(\overline{z}) = \begin{bmatrix}-c+\overline{z}_2 & \overline{z}_1\\ -\overline{z}_2 & a-\overline{z}_1\end{bmatrix}$ and will be evaluated at each equilibrium point in order to study its behavior.

The local behavior of the finite equilibrium points of the system (8) can be interpreted using the following Theorems and Corollaries:

Theorem 1 *Let p_1 and p_2 be the equilibrium points of the system (8), if $a < 0$ and $c > 0$, then p_1 is an attractor equilibrium point and p_2 is a saddle.*

Proof For p_1 to be an attractor equilibrium point the eigenvalues $\lambda_{1,2}$ must have negative signs, and in turn for p_2 to be a saddle the eigenvalues $\lambda_{3,4}$ must have opposite signs, then the necessary conditions are the following:

(i) $\lambda_1\lambda_2 = -ac > 0,$

(ii) $\lambda_1 + \lambda_2 = a - c < 0,$

(iii) $\lambda_3\lambda_4 = ac < 0.$

Since $a < 0$ and $c > 0$ the constraints $(i),(ii)$ and (iii) are satisfied, thus, p_1 behaves as an attractor equilibrium point and p_2 behaves like a saddle. □

Theorem 2 *Let p_1 and p_2 be the equilibrium points of the system (8), if $a < 0, c > 0$ and $|a| < |c|$, then p_1 is a repeller equilibrium point and p_2 is a saddle.*

Proof For p_1 to be a repeller equilibrium point the eigenvalues $\lambda_{1,2}$ must have positive signs and for p_2 to be a saddle the eigenvalues $\lambda_{3,4}$ must have opposite signs, then the necessary conditions are the following:

(i) $\lambda_1\lambda_2 = -ac > 0,$
(ii) $\lambda_1 + \lambda_2 = a - c > 0,$
(iii) $\lambda_3\lambda_4 = ac < 0.$

Since $a > 0$ and $c < 0$ the constraints $(i),(ii)$ and (iii) are satisfied if $|a|<|c|$, thus, p_1 behaves as a repeller equilibrium point and p_2 behaves as a saddle. □

Theorem 3 *Let* p_1 *and* p_2 *be the equilibrium points of the system (8), if* $a,c<0$ or $a,c>0$, *then* p_1 *is a saddle and* p_2 *is a center.*

Proof For p_1 to be a repeller equilibrium point the eigenvalues $\lambda_{1,2}$ must have opposite signs and for p_2 to be a center the eigenvalue $\lambda_{3,4}$ must be complex conjugates with a real part equal to zero and imaginary part different from zero, for this, the necessary conditions are the following:

(i) $\lambda_1\lambda_2 = -ac > 0,$
(ii) $Re[\lambda_{3,4}] = 0 \rightarrow Re[\lambda_3] = 0$ and $Re[\lambda_4] = 0,$
(iii) $Im[\lambda_{3,4}] \neq 0 \rightarrow Im[\lambda_3] = i\sqrt{ac} \neq 0$ and $Im[\lambda_4] = -i\sqrt{ac} \neq 0.$

If $a,c<0$ or $a,c>0$ the restrictions $(i),(ii)$ and (iii) are satisfied, thus, it is then proven, that if a,c have the same sign, p_1 is a saddle and p_2 is a center. □

Theorem 4 *Let* p_1 *and* p_2 *be the equilibrium points of the system (8), if* $a<0$, $c>0$ *and* $a=-c$, *then* p_1 *is a stable degenerate node and* p_2 *is saddle.*

Proof For p_1 to be a stable degenerate node the eigenvalues $\lambda_{1,2}$ must be equal and have negative signs, in turn for p_2 to be a saddle the eigenvalues $\lambda_{3,4}$ must have opposite signs so the necessary conditions are the following:

(i) $\lambda_1 = a = -c = \lambda_2,$
(ii) $\lambda_1\lambda_2 = -ac > 0,$
(iii) $\lambda_1 + \lambda_2 = a - c < 0,$
(iv) $\lambda_3\lambda_4 = ac < 0.$

The first condition is satisfied if $a=-c$ and since $a<0$, $c>0$ it is easy to check that $(i),(ii),(iii)$ and (iv) are fulfilled. Therefore, it is proven that if $a<0$, $c>0$ and also $a=-c$, then p_1 is a stable degenerate node and p_2 is a saddle. □

Theorem 5 *Let* p_1 *and* p_2 *be the equilibrium points of the system (8), if* $a>0$, $c<0$ *and* $a=-c$, *then* p_1 *is an unstable degenerate node and* p_2 *is saddle.*

Proof For p_1 to be an unstable degenerate node the eigenvalues $\lambda_{1,2}$ must be equal and have positive signs. In turn, for p_2 to be a saddle the eigenvalues $\lambda_{3,4}$ must have opposite signs, so the necessary conditions are the following:

(i) $\lambda_1 = a = -c = \lambda_2,$
(ii) $\lambda_1\lambda_2 = -ac > 0,$
(iii) $\lambda_1 + \lambda_2 = a - c > 0,$
(iv) $\lambda_3\lambda_4 = ac < 0.$

The first condition is satisfied if $a=-c$ and since $a>0$, $c<0$ it is easy to check that $(i),(ii),(iii)$ and (iv) are fulfilled. Therefore, it is proven that if $a>0$, $c<0$ and also $a=-c$, then p_1 is an unstable degenerate node and p_2 is a saddle. □

Corollary 1 *Let* p_1 *and* p_2 *be the equilibrium points of the system (8), if* $a,c>0$ *or* $a,c<0$ *and also* $a=c$, *then* p_1 *and* p_2 *have the behavior described in Theorem 3.*

Theorem 6 *Since the constraints of the parameters* $a=0$, $c\neq 0$ *or if* $a\neq 0, c=0$, *then there are infinitely stable equilibrium points and infinitely unstable equilibrium points.*

Proof In the case that $a=0$ and $c=0$, the system (8) takes the form

$$\dot{z}_1 = z_1(z_2 - c), \quad \dot{z}_2 = -z_2 z_1, \tag{10}$$

where there is an equilibrium point defined by $p_3 = (0, \bar{z}_2)$, that is, the axis z_2 is a line of infinite equilibrium points. By linearizing (10) at the point p_3, we obtain $A(\bar{z}) = \begin{bmatrix} \bar{z}_2 - c & 0 \\ -\bar{z}_2 & 0 \end{bmatrix}$, where the eigenvalues are $\lambda_1 = \bar{z}_2 - c$, and $\lambda_2 = 0.$

Since there exists the line $z_1 = 0$ with infinite equilibrium points and since that $\lambda_2 = 0$, then the *Theorem* 13 explains that there is a stable submanifold, an unstable submanifold and a tangent. The stable submanifold exists when $\lambda_1 = \overline{z}_2 - c < 0$, that is, when $\overline{z}_2 < c$, and the unstable submanifold exists when $\lambda_1 = \overline{z}_2 - c > 0$, that is, when $\overline{z}_2 > c$.

On the other hand, for the case that $a \neq 0$ and $c = 0$, the system (8) take the form

$$\dot{z}_1 = z_1 z_2, \quad \dot{z}_2 = z_2(a - z_1) \tag{11}$$

where there is an equilibrium point defined by $p_4 = (\overline{z}_1, 0)$, that is, the axis z_1 is a line of infinite equilibrium points. By linearizing (11) at the point p_4, we obtain $A(\overline{z}) = \begin{bmatrix} 0 & \overline{z}_1 \\ 0 & a - \overline{z}_1 \end{bmatrix}$, where the eigenvalues are $\lambda_1 = a - \overline{z}_1$ and $\lambda_2 = 0$.

Since there exists the line $z_2 = 0$ with infinitely equilibrium points and $\lambda_2 = 0$, using the *Theorem* 13, the stable submanifold exists when $\lambda_1 = a - \overline{z}_1 < 0$, that is, when $a < \overline{z}_1$, and the unstable submanifold when $\lambda_1 = a - \overline{z}_1 > 0$, that is, when $a > \overline{z}_1$. □

2.1.1 Resume of Finite Equilibrium Point Analysis

The theorems in the previous section demonstrate the behavior of the finite equilibria of the normalized system (8). This is a parametric analysis defined in the full range of parameters $a, c \in [\infty, \infty]$. The graphs contained within each of the quadrants of the plane a versus c, without the box and the four colors, represent the behavior of the finite equilibria per quadrant. The graphs within the quadrant boxes pointed with an arrow represent the behavior of the lines.

Fig. 2 shows how finite equilibria behave in the entire range of a, c. Generally quadrant I, $a > 0, c > 0$ is used as a model to represent the behavior of the species and specifically because the equilibrium point $(0,0)$ is one saddle and the other equilibrium (function of a, b) is a center.

All the behaviors corresponding to the linearization technique have been considered except for the parametric value $a = 0, c = 0$ where it is conjectured that the solution is only $x(t) = 0$.

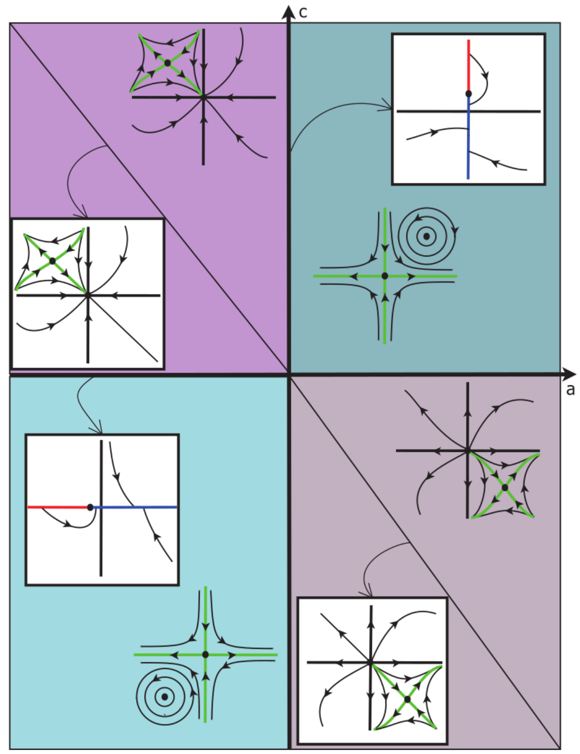

Fig. 2. Behaviors of Finite Equilibrium Points in Parameters Plane a vs. c.

2.2 *Analysis of Equilibrium Points in Infinity*

2.2.1 Analysis of the equilibrium points at infinity of the system (8)

Let X be the vector field associated with the system (8), then the expression for $\rho(X)$ on the local chart U_1 is obtained with the coordinates (u, v), which are defined by $z_1 = 1/v$, $z_2 = u/v$. From system (8) is obtained

$$P(z_1, z_2) = z_1(z_2 - c), \quad Q(z_1, z_2) = z_2(a - z_1), \tag{12}$$

substituting z_1, z_2 in the previous equation is obtained

$$P\left(\frac{1}{v}, \frac{u}{v}\right) = \frac{1}{v}\left(\frac{u}{v} - c\right), \quad Q\left(\frac{1}{v}, \frac{u}{v}\right) = \frac{u}{v}\left(a - \frac{1}{v}\right) \tag{13}$$

and by substituting (13) in (3) it takes the form

$$\dot{u} = -u - u^2 + uva + uvc, \quad \dot{v} = -uv + cv^2. \tag{14}$$

where the equilibrium points are $P_1 = (0,0)$, $P_2 = (-1,0)$ and $P_3 = (c/a, 1/a)$. The equilibrium point P_3 is discarded from the analysis since $\overline{v} \neq 0$.

Theorem 7 *Since the system (14), then* P_1 *and its an-*

tipodal $P_1^{'}$ behave as a saddle-node and P_2 behaves as a repeller node (or unstable node) while its antipodal $P_2^{'}$ behaves as an attractor node (or stable node).

Proof To study the behavior of the equilibrium points, the Linearization method must be applied to the system (14), obtaining that the matrix A of the linearized system is

$$A=\begin{bmatrix}-1-2\bar{u}+a\bar{v}+c\bar{v} & a\bar{u}+c\bar{u}\\ -\bar{v} & -\bar{u}+2c\bar{v}\end{bmatrix}.$$

Evaluating $P_1=(\bar{u},\bar{v})=(0,0)$ in A we obtain $A(P_1)=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}$, where the eigenvalues are $\lambda_1=-1$ and $\lambda_2=0$. Since one of the eigenvalues is zero, the infinite equilibrium point P_1 is semi-hyperbolic; therefore, its behavior is determined by the *Theorem* 12.

It is observed that the behavior of the equilibrium point does not depend on the values of the parameters a and c. Since $\lambda_1<0$ can be reduced to $\lambda_1>0$ by changing X to $-X$, then

$$\dot{u}=u+u^2-uva-uvc,\quad \dot{v}=uv-cv^2, \qquad (15)$$

for this system the linearized matrix evaluated at the equilibrium point P_1 becomes $A(P_1)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, where $\lambda_1=1$ and $\lambda_2=0$. For the system (15) to have the form of the system described in the *Theorem* 12, a linear transformation must be applied which is given by $u=V$ and $v=U$ staying

$$\dot{U}=UV-cU^2,\quad \dot{V}=V+V^2-aUV-cUV,$$

where $A(U,V)=UV-cU^2$, $B(U,V)=V^2-aUV-cUV$ and $\lambda>0$ are defined. Let $V=f(U)$ be the solution of the equation $V+V^2-aUV-cUV=0$, by solving the previous equation we obtain that $f(U)=0$ and $f(U)=-1+aU+cU$ are the solutions. Substituting the first and second solutions into $g(U)=A(U,f(U))$ is obtained $g_1(U)=-cU^2$ and $g_2(U)=-U+aU^2$, where it is observed that with $g_2(U)$ the *Theorem* 12 can't be applied, so the second solution is discarded. Then in $g_1(U)$ it is observed that $m=2$ and $a_m=-c$. Since m is even, part (iii) of *Theorem* 12 indicates that the equilibrium point P_1 behaves as a saddle-node.

Evaluating $P_2=(\bar{u},\bar{v})=(-1,0)$ in A we have $A(P_2)=\begin{bmatrix}1 & -a-c\\ 0 & 1\end{bmatrix}$, where the eigenvalues are $\lambda_1=\lambda_2=1$. In this case, the behavior of the equilibrium point does not depend on the values of the parameters a and c either. Since both eigenvalues are different from zero, we have a hyperbolic infinite equilibrium point, so its behavior is determined by the *Theorem* 11. Specifically in part (ii), $\lambda_{1,2}$ must be real with $|\lambda_2|\geq|\lambda_1|$, $\lambda_1\lambda_2>0$ and also $\lambda_1>0$. Since all the above conditions are met for $\lambda_{1,2}$ then the infinite equilibrium point $P_2^{'}$ behaves as a repeller node (or unstable node).

Finally, since the degree of the system (8) is even, that is, $d=2$, then the equilibrium point $P_1^{'}$ antipodal to P_1 which is located at infinity, it behaves as a saddle-node and the equilibrium point $P_2^{'}$ antipodal to P_2 behaves as an attractor node (or stable node). □

On the other hand, the expression for $\rho(X)$ on the local chart U_2 is obtained with the coordinates (u,v), which are defined by $z_1=u/v$, $z_2=1/v$. Substituting z_1,z_2 in (12) we obtain

$$P\left(\frac{u}{v},\frac{1}{v}\right)=\frac{u}{v}\left(\frac{1}{v}-c\right),\quad Q\left(\frac{u}{v},\frac{1}{v}\right)=\frac{1}{v}\left(a-\frac{u}{v}\right), \qquad (16)$$

substituting (16) into (4) gives

$$\dot{u}=u+u^2-auv-cuv,\quad \dot{v}=uv-av^2, \qquad (17)$$

where the equilibrium points are $P_4=(0,0)$, $P_5=(-1,0)$ and $P_6=(a/c,1/c)$. The equilibrium point P_6 is discarded from the analysis since $\bar{v}\neq 0$.

Theorem 8 *Since the system (17), then P_4 and its antipodal $P_4^{'}$ behave as a saddle-node and P_5 behaves as an attractor node (or stable node) while its antipodal $P_5^{'}$ behaves like a repeller node (or unstable node).*

Proof By applying the Linearization method to the system (17), then the matrix A of the linearized system is

$$A=\begin{bmatrix}1+2\bar{u}-a\bar{v}-c\bar{v} & -a\bar{u}-c\bar{u}\\ \bar{v} & \bar{u}-2a\bar{v}\end{bmatrix}.$$

Evaluating $P_4=(\bar{u},\bar{v})=(0,0)$ in A gives $A(P_4)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, where the eigenvalues are $\lambda_1=1$ and $\lambda_2=0$. Since one of the eigenvalues is zero, then the infinite equilibrium point P_4 is semi-hyperbolic, therefore its behavior is determined by the *Theorem* 12. It is observed that the behavior of the equilibrium point does not depend on the values of the parameters a and c.

For this case, a procedure similar to that developed in *Theorem* 7 is applied when analyzing the behavior of P_1, where for P_4 it is obtained that $m = 2$, $a_m = -a$ and given that m is even, part (iii) of *Theorem* 12 indicates that the equilibrium point behaves as a saddle-node.

Substituting $P_5 = (\bar{u}, \bar{v}) = (-1, 0)$ into A gives $A(P_5) = \begin{bmatrix} -1 & a+c \\ 0 & -1 \end{bmatrix}$, where the eigenvalues are $\lambda_3 = \lambda_4 = -1$. In this case, the behavior of the equilibrium point does not depend on the values of the parameters a and c either. Since both eigenvalues are different from zero, we have a hyperbolic equilibrium point at infinity and we have that in part (ii) of *Theorem* 12, $\lambda_{3,4}$ must be real with $|\lambda_4| \geq |\lambda_3|$, $\lambda_3\lambda_4 > 0$ and also $\lambda_3 < 0$. Since for $\lambda_{3,4}$ all the previous conditions are met then the infinite equilibrium point P_5 behaves as an attractor node (or stable node).

Finally, since the degree of the system (17) is even, that is, $d = 2$, then the equilibrium point $P_4^{'}$, antipodal to P_4, which is located at infinity, it behaves as a saddle-node and the equilibrium point $P_5^{'}$, antipodal to P_5, behaves as a repeller node (or unstable node). □

On the other hand, the expression for $\rho(X)$ on the local chart U_2 is obtained with the coordinates (u, v), which are defined by $z_1 = u/v$, $z_2 = 1/v$. Substituting z_1, z_2 in (12) we obtain

$$P\left(\frac{u}{v}, \frac{1}{v}\right) = \frac{u}{v}\left(\frac{1}{v} - c\right), \quad Q\left(\frac{u}{v}, \frac{1}{v}\right) = \frac{1}{v}\left(a - \frac{u}{v}\right), \qquad (16)$$

substituting (16) into (4) gives

$$\dot{u} = u + u^2 - auv - cuv, \quad \dot{v} = uv - av^2, \qquad (17)$$

where the equilibrium points are $P_4 = (0, 0)$, $P_5 = (-1, 0)$ and $P_6 = (a/c, 1/c)$. The equilibrium point P_6 is discarded from the analysis since $\bar{v} \neq 0$.

2.2.2 Analysis of the equilibrium points at infinity of the system (10)

Let X_1 be the vector field associated with the system (10). Then the expression for $\rho(X_1)$ on the local chart U_1 is obtained with the coordinates (u, v), which are defined by $z_1 = 1/v$, $z_2 = u/v$. From (10) we have that $P(z_1, z_2) = z_1(z_2 - c)$, $Q(z_1, z_2) = -z_2 z_1$, where this system has z_1 as a common factor, for which a linear transformation defined by $z_1 dt = ds$ is applied to facilitate the calculations, obtaining

$$\frac{dz_1}{ds} = P(z_1, z_2) = z_2 - c, \quad \frac{dz_2}{ds} = Q(z_1, z_2) = -z_2. \qquad (18)$$

Remark 1 The orbits of the system (10) are obtained from the orbits of the system (18) as follows:
- The system (10) has the line $z_1 = 0$ of equilibrium points, which the system (18) does not have.
- The system (10) in the half-plane $z_1 > 0$ has the same orbits as the system (18) in the same direction but with different speeds.
- The system (10) in the half-plane $z_1 < 0$ has the same orbits as the system (18) in the opposite direction with different speeds.

Substituting $z_1 = 1/v$, $z_2 = u/v$ in the system (18)

$$P\left(\frac{1}{v}, \frac{u}{v}\right) = \frac{u}{v} - c, \quad Q\left(\frac{1}{v}, \frac{u}{v}\right) = -\frac{u}{v}, \qquad (19)$$

and substituting (19) into (3), we obtain

$$\dot{u} = -u - u^2 + uvc, \quad \dot{v} = -uv + cv^2, \qquad (20)$$

where the equilibrium points are $P_7 = (0, 0)$, $P_8 = (-1, 0)$.

Theorem 9 *Since the system (20), then* P_7 *and its antipodal* $P_7^{'}$ *behave as a saddle-node and* P_8 *behave as a repeller node (or unstable node), while it's antipodal* $P_8^{'}$ *behave as an atractor node (or stable node).*

Proof Applying the Linearization method to the system (20), we obtain that the matrix A of the linearized system is

$$A=\begin{bmatrix}-1-2\bar{u}+c\bar{v} & c\bar{u}\\ \bar{v} & -\bar{u}+2c\bar{v}\end{bmatrix}.$$

Evaluating $P_7=(\bar{u},\bar{v})=(0,0)$ in A gives $A(P_7)=\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}$, where the eigenvalues are $\lambda_1=-1$ and $\lambda_2=0$. Since one of the eigenvalues is zero, then the infinite equilibrium point P_7 is semi-hyperbolic, therefore its behavior is determined by the *Theorem* 12.

In this case, the same procedure developed in *Theorem* 7 will be applied when analyzing the behavior of P_1, obtaining for P_7 that $m=2$, $a_m=-c$ and given that m is even, so the equilibrium point behaves as a saddle-node.

Now, $P_8=(\bar{u},\bar{v})=(-1,0)$ is evaluated in A obtaining $A(P_8)=\begin{bmatrix}1 & -c\\ 0 & 1\end{bmatrix}$, where the eigenvalues are $\lambda_1=\lambda_2=1$. Since both eigenvalues are different from zero, we have that P_8 is a hyperbolic equilibrium point and applying part (ii) of *Theorem* 11, $\lambda_{1,2}$ must be real with $|\lambda_2|\geq|\lambda_1|$, $\lambda_1\lambda_2>0$ and also $\lambda_1>0$. Since all the above conditions are met for $\lambda_{1,2}$ then the infinite equilibrium point P_8 behaves as a repeller node (or unstable node).

Finally, since the degree of the system (10) is even, that is, $d=2$, then the equilibrium point $P_7^{'}$ antipodal to P_7 which is located at infinity, behaves as a saddle-node and the equilibrium point $P_8^{'}$ antipodal to P_8 behaves as an attractor node (or stable node).

2.2.3 Analysis of the equilibrium points at Infinity of the system (11)

Let X_2 be the vector field associated with the system (11), then the expression for $\rho(X_2)$ on the local chart U_2 is obtained with the coordinates (u,v), defined for $z_1=u/v$, $z_2=1/v$.

From (11) we have $P(z_1,z_2)=z_1z_2$, $Q(z_1,z_2)=z_2(a-z_1)$, where this system has z_2 as a common factor, for which the linear transformation $z_2dt=ds$ is applied to facilitate the calculations, obtaining

$$\frac{dz_1}{ds}=P(z_1,z_2)=z_1,\ \frac{dz_2}{ds}=Q(z_1,z_2)=(a-z_1), \quad (21)$$

Remark 2. The orbits of the system (11) are obtained from the orbits of the system (21) as follows:

- The system (11) has the line $z_2=0$ of equilibrium points, which the system (21) does not have.
- The system (11) in the half-plane $z_2>0$ has the same orbits as the system (21) in the same direction but with different speeds.
- The system (11) in the half-plane $z_2<0$ has the same orbits as the system (21) in the opposite direction with different speeds.

Substituting $z_1=u/v$, $z_2=1/v$ in the system (21)

$$P\left(\frac{u}{v},\frac{1}{v}\right)=\frac{u}{v},\quad Q\left(\frac{u}{v},\frac{1}{v}\right)=a-\frac{u}{v}, \quad (22)$$

and by substituting (22) in (4) we obtain

$$\dot{u}=u+u^2-uva,\quad \dot{v}=uv-av^2, \quad (23)$$

where the equilibrium points are $P_9=(0,0)$, $P_{10}=(-1,0)$.

Theorem 10 *Since the system (23), the equilibrium point P_9 and its antipodal $P_9^{'}$ behave like a saddle-node the equilibrium point P_{10} behaves as an attractor node (or stable node), while it's antipodal $P_{10}^{'}$ behaves as repeller node (or unstable node).*

Proof By applying the Linearization method to the system (23), we obtain that the matrix A is

$$A=\begin{bmatrix}1+2\bar{u}-a\bar{v} & -a\bar{u}\\ \bar{v} & \bar{u}-2a\bar{v}\end{bmatrix}.$$

Evaluating $P_9=(\bar{u},\bar{v})=(0,0)$ in A gives $A(P_9)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, where the eigenvalues are $\lambda_1=1$ and $\lambda_2=0$. Given that one of the eigenvalues is zero, the equilibrium point P_9 is a semi-hyperbolic equilibrium point and applying the same procedure developed in *Theorem* 7 when analyzing the behavior of P_1, for P_9 we obtain that $m=2$, $a_m=-a$ and since m is even, the equilibrium point behaves as a saddle-node.

Now, $P_{10} = (\bar{u}, \bar{v}) = (-1, 0)$ is evaluated in A to study the behavior of the equilibrium point $A(P_{10}) = \begin{bmatrix} -1 & a \\ 0 & -1 \end{bmatrix}$, where the eigenvalues are $\lambda_1 = \lambda_2 = -1$. Since both eigenvalues are different from zero, P_{10} is a hyperbolic equilibrium point and according to part (ii) of *Theorem 12*, $\lambda_{1,2}$ must be real with $|\lambda_2| \geq |\lambda_1|$, $\lambda_1\lambda_2 > 0$ and also $\lambda_1 < 0$. Since all the above conditions are met for $\lambda_{1,2}$ then the infinite equilibrium point P_{10} behaves as an attractor node (or stable node).

Finally, since the degree of the system (11) is even, that is, $d = 2$, then the equilibrium point $P_9^{'}$ antipodal to P_9 which is located at infinity , behaves as a saddle-node and the equilibrium point $P_{10}^{'}$ antipodal to P_{10} behaves as a repeller node (or unstable node).

Tables 1 and 2 summarize the qualitative behavior of the system (8) that were obtained in *Theorems* 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, in addition to p_1, p_2, p_3 and p_4 are the finite equilibrium points; P_1, $P_1^{'}$, P_2, P_4, $P_4^{'}$, P_5, P_7, $P_7^{'}$, P_8, $P_8^{'}$, P_9, $P_9^{'}$, P_{10} and $P_{10}^{'}$ are equilibrium points at infinity. Also, SN: stable node, UN: unstable node, S: saddle, C: center, UDN: unstable degenerate node, SDN: stable degenerate node, S-N: saddle-node.

Table 1: Summary of Finite and Infinity Equilibrium Points

Qualitative Map		p_1	p_2	P_1	$P_1^{'}$	P_2	P_5	P_4	$P_4^{'}$
1 (Fig.3(a))	a<0,c>0	SN	S	S-N	S-N	UN	SN	S-N	S-N
2 (Fig.3(b))	a>0,c<0	UN	S	S-N	S-N	UN	SN	S-N	S-N
3 (Fig.3(c))	a>0,c>0	S	C	S-N	S-N	UN	SN	S-N	S-N
4 (Fig.3(d))	a<0,c<0	S	C	S-N	S-N	UN	SN	S-N	S-N
5 (Fig.3(e))	a<0,c>0 a=-c	SDN	S	S-N	S-N	UN	SN	S-N	S-N
6 (Fig.3(f))	a>0,c<0 a=-c	ND	S	S-N	S-N	UN	SN	S-N	S-N

Table 2: Summary of Equilibrium Points

Qualitative Map		p_3	P_7	$P_7^{'}$	P_8	$P_8^{'}$
7 (Fig.4(g)) 8 (Fig.4(h))	a=0,c>0 a=0,c<0	z_2<c Stable Manifold z_2>c Unstable Manifold	S-N	S-N	UN	SN
Qualitative Map		**p_4**	**P_9**	**$P_9^{'}$**	**P_{10}**	**$P_{10}^{'}$**
9 (Fig.4(i)) 10 (Fig.4(j))	a>0,c=0 a<0,c=0	z_1<c Stable Manifold z_1>c Unstable Manifold	S-N	S-N	SN	UN

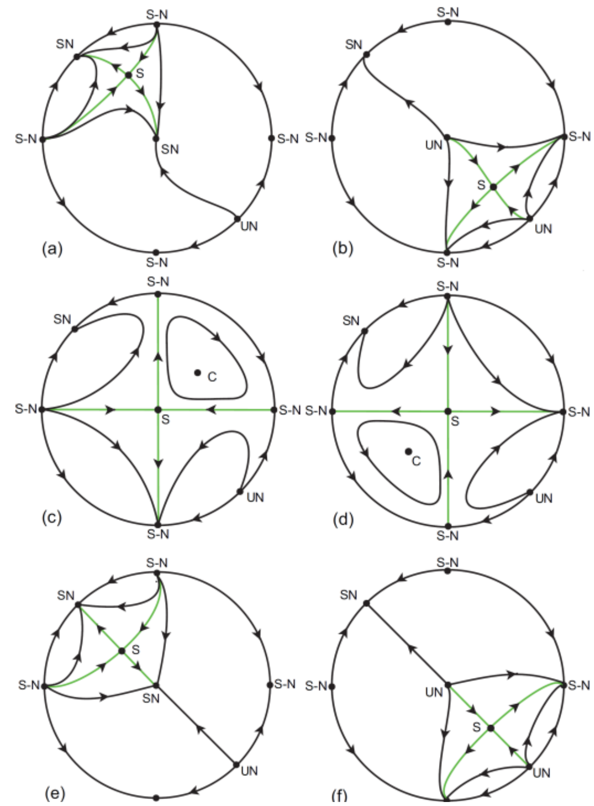

Fig. 3. Poincaré Disk of system (8).

In the qualitative maps 1, 2, 3, 4, 5 and 6 of Tables 1 and 2, the system (8) has 2 finite equilibria and 6 equilibrium points at infinity as seen in the Fig. 3, while in the qualitative maps 7, 8, 9 and 10 there are infinitely many finite equilibrium points and 4 equilibrium points at infinity, as seen in Fig. 4 and 5.

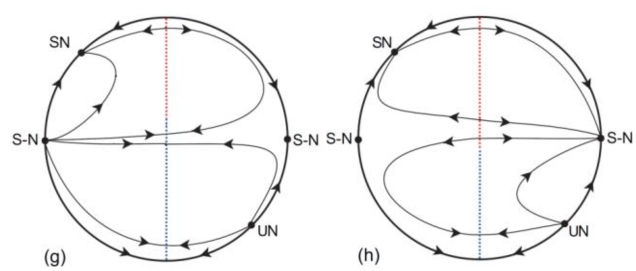

Fig. 4. Poincaré Disk of system (10)

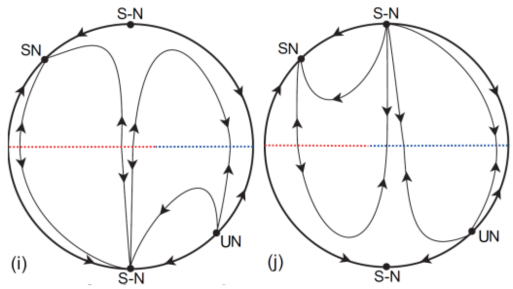

Fig. 5. Poincaré Disk of system (11).

5 Analysis of Trajectories

The procedure allows obtaining information regarding the finite equilibrium points and their vicinities, the equilibrium points at infinity and their vicinities, but there is no information about the flow Φ that exists outside the equilibrium points and their vicinities. To explain the behavior of the flow, in this research it has been decided to present the following conjecture:

Conjecture 1. The flow Φ is continuous and differentiable for any path ϕ and for any initial condition.

This conjecture is based on the text of Hartman (1973), especially in the chapter referring to the *Continuous Dependence of the Solutions on the Initial Conditions and Parameters* and specifically the *Theorems* 2.1 and 3.1, pp 94 and 95 respectively, which refer to continuity and differentiability.

If the conjecture is true and the theorems are sufficient, the flow Φ is homogeneous throughout the space $\mathbb{R}^2$ without taking into account the set of equilibrium points and invariant manifold, in words simple, the flow will not contain irregularities.

6 Conclusions

After applying the suggested methodology to the Predator-Prey model, interesting results were obtained regarding qualitative behavior that can be summarized as:

- The model has two finite equilibrium points p_1, p_2 as long as none of the parameters a, c is equal to zero; its behavior is demonstrated in *Theorems* 1, 2, 3, 4 and 5.
- If one of the parameters a, c is equal to zero, a straight line of finite equilibrium points appears in the model, its behavior is demonstrated in *Theorem* 6.
- The model has six equilibrium points at infinity as long as none of the parameters a, c is equal to zero; its behavior is demonstrated in *Theorems* 7 and 8.
- If one of the parameters a, c is equal to zero, the system has four equilibrium points at infinity; its behavior is demonstrated in *Theorems* 9 and 10.
- After applying *Poincaré Compactification*, ten *Poincaré Disks* were obtained that describe the global behavior of the system for all parameter values, except when both a, c are equal to zero.

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7 Appendix

Theorem 11 (Hyperbolic Singular Point Theorem). *Let $(0,0)$ be a singular point isolated from a vector field X given by*

$$\begin{aligned} \dot{x} &= ax + by + A(x,y), \\ \dot{y} &= cx + dy + B(x,y), \end{aligned} \tag{24}$$

where A and B are analytic functions in a vicinity of the origin with $A(0,0) = B(0,0) = DA(0,0) = DB(0,0) = 0$, DA and DB are the derivatives of the functions A and B respectively. Let λ_1 and λ_2 be eigenvalues of the linear part $DX(0)$ of the system at the origin. So the following statements hold.

(i) If λ_1 and λ_2 are real and $\lambda_1\lambda_2 < 0$, then $(0,0)$ is a chair (see Fig. 6(a)). If we denote by E_1 and E_2 the eigenspaces of λ_1 and λ_2 respectively, then two invariant analytical curves can be found, tangent to E_1 and E_2 at 0 respectively, at one of the points they are attracted towards the origin, and at one of the points they are repelled from the origin.

(ii) If λ_1 and λ_2 are real with $|\lambda_1| \geq |\lambda_2|$ and $\lambda_1\lambda_2 > 0$, then $(0,0)$ is a node (see Fig. 6(b)). If $\lambda_1 > 0$ (respectively $\lambda_1 < 0$) then it is a repeller or unstable node (respectively attractor or stable node).

(iii) If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ with $\alpha, \beta \neq 0$, then $(0,0)$ is a "strong" focus (see Fig. 6(c)). If $\alpha > 0$ (respectively $\alpha < 0$), it is a repeller focus or unstable focus (respectively attractor focus or stable focus).

(iv) If $\lambda_1 = i\beta$ and $\lambda_2 = -i\beta$ with $\beta \neq 0$, then $(0,0)$ is a linear center, a "weak" focus or center (see Fig. 6(d)).

(v) If $\lambda_1 = \lambda_2$, the we have an equilibrium point that behaves like a degenerate node, (see figure 6(e)). If $\lambda_1 > 0$ (respectively $\lambda_1 < 0$) then the equilibrium point behaves as a unestable degenerate node (respectively a stable degenerate node).

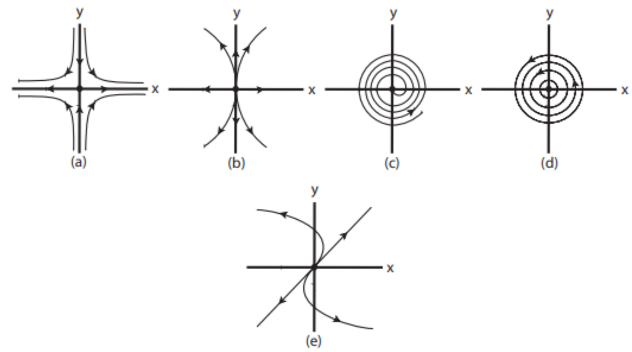

Fig. 6. Phase-Portraits of Hyperbolic Equilibrium Points

Theorem 12 (Semi-Hyperbolic Singular Point Theorem). *Let $(0,0)$ be a singular point isolated from a vector field X given by*

$$\begin{aligned} \dot{x} &= A(x,y), \\ \dot{y} &= \lambda y + B(x,y), \end{aligned} \tag{25}$$

where A and B are analytic functions in a vicinity of the origin with $A(0,0) = B(0,0) = DA(0,0) = DB(0,0) = 0$, DA and DB are the derivatives of the functions A and B respectively. Let $y = f(x)$ be the solution of the equation $\lambda y + B(x,y) = 0$ in a vicinity of the point $(0,0)$, and assuming that the function $g(x) = A(x, f(x))$ has the expression $g(x) = a_m x^m + o(x^m)$, where $m \geq 2$ and $a_m \neq 0$. Then there will always exist an invariant analytical curve, called strong unstable manifold, tangent to 0 on the y-axis, in which X is analytically conjugate to $\dot{y} = \lambda y$; represents a behavior repeller since $\lambda > 0$. Furthermore, the following statements hold up.

(i) If m is odd and also $a_m < 0$, then $(0,0)$ is a topological saddle (see Fig. 7(a)).

(ii) If m is odd and also $a_m < 0$, then $(0,0)$ is a topological unstablenode (see Fig. 7(b)).

(iii) If m is even, then $(0,0)$ is a chair node (S-N), (see Fig. 7(c)).

Remark 3. The case $\lambda < 0$ can be reduced to $\lambda > 0$ by changing X to $-X$. Furthermore, for the *Theorem* 12 in the case that $g(x) = A(x, f(x))$ is identically zero, and then there will be an analytical curve consisting of equilibrium points.

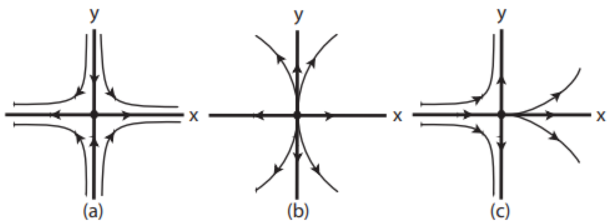

Fig. 7. Phase-Portraits of Semi-Hyperbolic Equilibrium Points.

Theorem 13 (Normally Hyperbolic Invariant Maniflod). *Let Λ be a normally hyperbolic submanifold of equilibrium points for ϕ_t. Then there exist smooth manifolds along stable and unstable Λ that are tangent to $E^s \oplus T\Lambda$ and $E^u \oplus T\Lambda$, respectively. Furthermore, Λ and the stable and unstable manifolds are permanent under small flow perturbations.*

Recibido: *3 de octubre de 2024*

Aceptado: *10 de diciembre de 2024*

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