

Algebraic design of two-degree-of-freedom PID controllers for FOPTD systems

Diseño algebraico de controladores PID de dos grados de libertad para sistemas FOPTD

Teppa-Garran, Pedro^{1*}; Caraballo, Luis²; Garcia, Germain³

¹Departamento de Gestión de Proyectos y Sistemas, Universidad Metropolitana, Caracas, Venezuela.

²Escuela de Ingeniería Eléctrica, Universidad Metropolitana, Caracas, Venezuela.

³CNRS, LAAS, 7 Avenue du Colonel Roche, Toulouse, France, Université de Toulouse, INSA, Toulouse, France.

*teppa@unimet.edu.ve

Abstract

This work proposes a model-based tuning method for two-degree-of-freedom PID controllers founded on a polynomial approach and fundamental notions of control theory for first-order plus time-delay (FOPTD) systems. The technique achieves rejection of constant disturbance signals through a time response that asymptotically approaches zero with a specified overshoot and settling time. Likewise, tracking with zero steady-state error of persistent reference inputs such as step, ramp, or parabola is also attained. Another contribution of the method is that it allows tuning the four PID controller parameters. Several examples show the ease of implementation of the technique, its effectiveness, and how it can be extended to systems other than FOPTD.

Keywords: PID, Two-degree-of-freedom PID, Time delay systems, FOPTD, Tracking, Disturbance Rejection, Heat flow process, Coupled tank system, United Nations SDG 9.

Resumen

Este trabajo propone un método basado en modelos para la sintonización de controladores PID de dos grados de libertad para sistemas de primer orden con retardo en el tiempo (FOPTD) fundamentado en un enfoque polinómico y nociones básicas de la teoría de control. La técnica logra el rechazo de señales de perturbación constante mediante una respuesta temporal que tiende asintóticamente a cero con un sobrepico y un tiempo de establecimiento especificados. Asimismo, se logra un seguimiento con error de estado estacionario cero de entradas de referencia persistentes, como el escalón, la rampa o la parábola. Otra contribución del método es que permite ajustar los cuatro parámetros del controlador PID. Varios ejemplos muestran la facilidad de implementación de la técnica, su eficacia y su extensión a sistemas distintos de los FOPTD.

Palabras clave: PID, PID de dos grados de libertad, Sistemas con tiempo de retardo, FOPTD, Seguimiento, Rechazo de perturbaciones, Proceso de flujo de calor, Sistema de tanques acoplados, ODS 9 de las Naciones Unidas.

1 Introduction

Two fundamental problems arise in the design of control systems: tracking a reference input and rejecting disturbance signals. A basic notion of control theory is that a physical variable can behave in a prescribed manner by using the difference between a desired reference value and the actual output value until the two are matched. This notion results in the classic feedback control loop, where the control signal is generated using the difference of signals indicated above; this scheme is also called a one-degree-of-freedom

(1-DoF) controller. The degree of freedom of a control system is defined as the number of closed-loop transfer functions that can be independently adjusted (Teppa-Garran et al., 2023).

Time-delay control systems are present in numerous industrial applications, such as chemical engineering, biochemistry, aerospace, and power generation, to name just a few (Gu and Niculescu, 2003; Birs et al., 2019). Even if it is not a natural component of the process, actuators and sensors in a control system introduce time delays in its operation (Richard, 2003; Ai et al., 2016). The main reason

for time delays in industrial processes is transporting materials or energy in long pipelines (Bresch-Pietri et al., 2014) or data traffic in communication networks (Ariba et al., 2009). Time delay is known to degrade process performance. From the frequency domain point of view, delay introduces an additional lag in the process phase. This results in lower phases, gain margins, and the possibility of losing stability. This has led to the study of the stability of time delay systems by the Lyapunov and Krasovskii analysis (Hale and Lunel, 2013) and variants within this general analysis procedure, such as the Wirtinger-based inequality (Seuret and Gouaisbaut, 2013), and the Bessel-Legendre inequalities (Zhang et al., 2019, 2022).

As a special case of time-delay systems, many open-loop industrial processes of practical interest can be effectively modeled by a low-order transfer function cascade with a time delay. First-order plus time-delay (FOPTD) systems are the most commonly employed process model in control design (O'Dwyer, 2009).

The proportional-integral-derivative (PID) controller is the industry's most widely used control strategy nowadays (Åström and Hagglund, 1995; Desborough and Miller, 2002). Its success is attributed to its simple structure, the meaning of its three parameters, its easy understanding by technical personnel, and because it provides stability and fast responses for a wide range of operating conditions.

However, despite its wide use, many poorly tuned PID controllers are found at the industrial level (Lee et al., 2015). In controlling systems with time delay, the performance of PID controllers also has limitations. Several factors can be listed to explain this situation, such as nonlinearities, uncertainties, external disturbances, variable loads, etc. (Liu and Daley, 2001). Still, one essential reason is that a 1-DoF controller must achieve a compromise between tracking a reference input and disturbance attenuation (Vilanova et al., 2011; Teppa-Garran et al., 2023). In a two-degree-of-freedom (2-DoF) controller, the reference input and the controlled output are processed independently to generate the control signal. This additional degree of freedom allows the two fundamental control problems mentioned at the beginning to be solved separately. The need to satisfactorily solve both issues motivated the introduction of two-degree-of-freedom PID controllers (2-DoF PID) (Araki, 1985; Araki and Taguchi, 2003). Since then, many methods for tuning their parameters have been proposed, including the internal model control (Mamat, 2013; Jin and Liu, 2014). The gain-phase margin (Xing et al., 2006). The maximum sensitivity function (Alfaro et al., 2010). The fractional order PID controller (Sharma et al., 2015). The desired dynamic equation (Zhang et al., 2010). Combining the desired dynamic equation with the generalized frequency method (Wang et al., 2018) and fuzzy control (Bi, 2020).

Most of these methods rely on complex mathematical notions that undermine the simplicity of understanding the PID operation by industrial technical personnel. The main contribution of our study is to propose a methodology for

tuning the four parameters (K_p, K_i, K_d, τ_d) of a 2-DoF PID controller for a FOPTD plant using a straightforward algebraic approach based on elementary notions of control theory. To this end, a 2-DoF general control system architecture defined by (Araqui and Taguchi, 2003) is employed, which consists of a feedforward and a serial PID controller. The disturbance attenuation problem is solved using a dominant pole guarantee criterion proposed in (Persson and Åström, 1992) and used in various applications, for instance (Teppa-Garran and Garcia, 2017; Teppa-Garran and Vasquez, 2020). Applying the dominant pole guarantee criterion, the control system's response to any constant disturbance signal tends asymptotically to zero, exhibiting a desired overshoot and settling time. On the other hand, the problem of tracking persistent signals of the step, ramp, or parabola type is solved using a zero assignment criterion for the closed-loop transfer function. Several numerical examples show the proposed methodology's effectiveness and ease of implementation.

The results of this work promote innovation in the control of industrial processes by improving the tuning of PID controllers in a 2-DoF control system architecture and, in this way, contribute to SDG 9 of the United Nations.

2 Problem formulation and basic notions

The control scheme in Fig. 1 shows a general two-degree-of-freedom architecture (Araqui and Taguchi, 2003). The signal $r(t): \mathbb{R}^+ \rightarrow \mathbb{R}$ is the reference input, $d(t): \mathbb{R}^+ \rightarrow \mathbb{R}$ is a disturbance signal, $y(t): \mathbb{R}^+ \rightarrow \mathbb{R}$ corresponds to the controlled output, $u(t): \mathbb{R}^+ \rightarrow \mathbb{R}$ is the control signal, and t represents the independent time variable. The controller consists of two PID compensators represented by the transfer functions $G_1(s)$ and $G_2(s)$. The tracking error is defined by

$$e(t) = r(t) - y(t) \quad (1)$$

There are two closed-loop transfer functions in Fig. 1. One from d to y and the other from r to y ; they are given respectively by the expressions.

$$G_{yd}(s) = \frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + G_1(s)G_p(s)} \quad (2)$$

$$G_{yr}(s) = \frac{Y(s)}{R(s)} = \frac{[G_1(s) + G_2(s)]G_p(s)}{1 + G_1(s)G_p(s)} \quad (3)$$

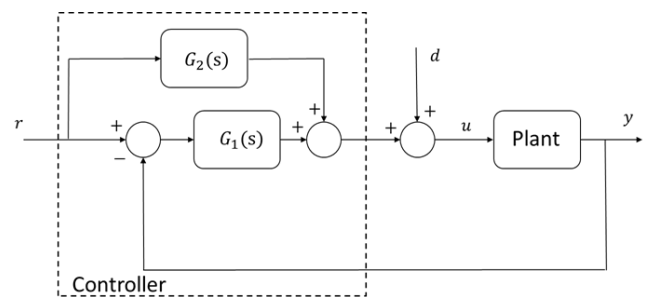


Figure 1. Two-degree-of-freedom (2-DoF) control system.

2.1 Plant model

The plant is modeled by a first-order system with transport delay θ_d (FOPTD) characterized by the transfer function.

$$\frac{Y(s)}{U(s)} = \frac{K}{Ts + 1} e^{-\theta_d s} \quad (4)$$

In this work, the transport delay is approximated in two different ways. (i) Using the denominator Taylor series expansion ($e^{-\theta_d s} \approx 1/(1 + \theta_d s)$) (Hanta and Procháska, 2009). The plant model takes the form

$$G_p(s) = \frac{b_0}{s^2 + a_1 s + a_0} \quad (5)$$

Where

$$b_0 = K/T\theta_d, a_1 = (T + \theta_d)/T\theta_d, a_0 = 1/T\theta_d$$

(ii) Using a first-order Padé model ($e^{-\theta_d s} \approx (2 - \theta_d s)/(2 + \theta_d s)$) (Hanta and Procháska, 2009) allows us to obtain the following transfer function for the plant

$$G_p(s) = \frac{-b_1 s + b_0}{s^2 + a_1 s + a_0} \quad (6)$$

Where

$$b_1 = K/T, b_0 = 2K/T\theta_d, a_1 = (2T + \theta_d)/T\theta_d, a_0 = 2/T\theta_d$$

Equations (5, 6) will be used to design the PID controllers $G_1(s)$ and $G_2(s)$ of Fig. 1.

2.2 PID controller

The mathematical model of a PID controller is given by

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau) d\tau + K_d \frac{d}{dt} e(t) \quad (7)$$

Or, in the Laplace domain, through the equation

$$U(s) = \left(K_p + \frac{K_i}{s} + K_d s \right) E(s) \quad (8)$$

The error signal $e(t)$ is used to generate the proportional, integral, and derivative actions, which are combined to form the control signal $u(t)$. The PID controller parameters are the proportional K_p , the integral K_i , and the derivative K_d constants. These constants must be tuned to meet the design requirements, and several empirical tuning rules have been formulated for this purpose. A non-exhaustive list is given in (Teppa-Garran et al., 2021). These easy-to-implement rules provide tuning methods for PID controller parameters that often do not result in the best constant settings (Lee et al., 2014). For this reason, optimal tuning methods have been proposed: based on genetic algorithms (El-Deen et al., 2015; Gunawan et al., 2018), computer-assisted (Teppa-Garran et al., 2021; Teppa-Garran and El Gharib, 2024), or based on the linear-quadratic regulator (Teppa-Garran et al., 2025a).

In practical applications, the pure derivative action in (7) is never used, due to the derivative kick and the amplification of measurement noise (Atherton and Majhi,

1999; Zhu, 2009). For this reason, the derivative term is cascaded with a first-order low-pass filter, resulting in equation (8) becoming,

$$U(s) = \left(K_p + \frac{K_i}{s} + \frac{K_d s}{\tau_d s + 1} \right) E(s) \quad (9)$$

All the rules (empirical and optimal) mentioned previously consider only the tuning of the three constants K_p , K_i , and K_d , but not the term τ_d . Subsequent simulations based on predefined ranges are usually used to determine the latter. For example, the following interval is proposed in (Goodwin et al., 2001).

$$0.1(K_d/K_p) \leq \tau_d \leq 0.2(K_d/K_p) \quad (10)$$

In this study, the transfer functions of the PID controllers $G_1(s)$ and $G_2(s)$ are modeled by an equation of the form (9). Another significant contribution of this work is the proposal of a method that directly adjusts the four parameters of the PID controller.

2.3 Problem formulation

This work simultaneously aims to solve the two fundamental control problems mentioned in the introduction. To this end, the following problem is formulated.

Problem 1: Given a plant model in the form (5) or (6), determine the four parameters in the controllers $G_1(s)$ and $G_2(s)$ modeled through (9) so that the controlled output $y(t)$ in the control system of Fig. 1 satisfies the following two conditions: (i) It approaches asymptotically to zero exhibiting a desired overshoot (OS) and settling time (T_s) when a constant disturbance signal $d(t)$ is applied. (ii) It tracks with zero steady-state error a reference input $r(t)$ of step, ramp, or parabola type.

3 Problem solution

This section develops the method for synthesizing the controllers $G_1(s)$ and $G_2(s)$ to solve problem 1. The transfer function of $G_1(s)$ is given by

$$G_1(s) = K_{p1} + \frac{K_{i1}}{s} + \frac{K_{d1} s}{\tau_{d1} s + 1} \quad (11)$$

And the transfer function of $G_2(s)$ through

$$G_2(s) = K_{p2} + \frac{K_{i2}}{s} + \frac{K_{d2} s}{\tau_{d2} s + 1} \quad (12)$$

In this work, it is assumed that the following condition is satisfied in (11) and (12)

$$\tau_d = \tau_{d1} = \tau_{d2} \quad (13)$$

Let the controller $G_{12}(s)$ be defined as

$$G_{12}(s) = G_1(s) + G_2(s) = K_p + \frac{K_i}{s} + \frac{K_d s}{\tau_d s + 1} \quad (14)$$

Where $K_p = K_{p1} + K_{p2}$, $K_i = K_{i1} + K_{i2}$ and $K_d = K_{d1} + K_{d2}$. A fundamental result for this study is shown below.

Theorem 1: Any controller of the form

$$G_c(s) = \frac{As^2 + Bs + C}{Ds^2 + Es} \quad (15)$$

It is identical to a PID controller expressed by the equation (9), where

$$\begin{aligned} K_p &= \frac{BE - CD}{E^2} \\ K_i &= \frac{C}{E} \\ K_d &= \frac{AE^2 - BDE + CD^2}{E^2} \\ \tau_d &= \frac{D}{E} \end{aligned} \quad (16)$$

Proof: Equation (9) is rewritten as

$$\frac{(K_{d1} + K_{p1}\tau_d)s^2 + (K_{p1} + K_{i1}\tau_d)s + K_{i1}}{\tau_d s^2 + s} \quad (17)$$

And (15) as

$$\frac{\frac{A}{E}s^2 + \frac{B}{E}s + \frac{C}{E}}{\frac{D}{E}s^2 + s} \quad (18)$$

The result is established by equating (17) and (18).

Considering Theorem 1 and the condition (13), the transfer functions of the controllers $G_1(s)$, $G_2(s)$, and $G_{12}(s)$ can be expressed by equations (19), (20), and (21), respectively.

$$G_1(s) = \frac{e_2 s^2 + e_1 s + e_0}{d_2 s^2 + d_1 s} \quad (19)$$

$$G_2(s) = \frac{f_2 s^2 + f_1 s + f_0}{d_2 s^2 + d_1 s} \quad (20)$$

$$G_{12}(s) = G_1(s) + G_2(s) = \frac{n_2 s^2 + n_1 s + n_0}{d_2 s^2 + d_1 s} \quad (21)$$

Where $n_2 = e_2 + f_2$, $n_1 = e_1 + f_1$, and $n_0 = e_0 + f_0$.

3.1 Synthesis of controller $G_1(s)$

The PID controller in (11) with condition (13) expressed in the form (19) is to be designed such that the time component of the controlled output $y(t)$ in response to a constant disturbance $d(t)$ tends asymptotically to zero with a desired overshoot (OS) and settling time (T_s). This objective is best appreciated in the block diagram of Fig. 2, which results from the simplification of the diagram in Fig. 1 by setting $r(t) = 0$.

A dominant pole guarantee criterion specifies the closed-loop performance requirements of the time response to a constant disturbance signal. This criterion allows the

construction of a desired closed-loop polynomial consisting of a pair of complex conjugate poles that will dominate the time response dynamics, with the remaining poles making a negligible contribution. The dominance of the complex conjugate pole pair $s_{1,2} = -\alpha \pm j\beta$ requires that the ratio of the real parts of the other poles to $-\alpha$ exceeds a factor λ (λ is typically taken as 3 to 10 times). Thus, the other poles are positioned to the left of the vertical line $s = -\lambda\alpha$. In this work, the desired closed-loop polynomial is expressed as

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + \lambda\zeta\omega_n)^2 \quad (22)$$

The damping coefficient ζ and the natural frequency ω_n of the dominant poles are determined in terms of the desired overshoot OS and settling time T_s by making (Dorf and Bishop, 2017).

$$OS = e^{(-\zeta\pi/\sqrt{1-\zeta^2})} \Rightarrow \zeta = \frac{1}{\sqrt{1 + \left(\frac{\pi}{\ln(OS)}\right)^2}} \quad (23)$$

$$T_s = 4/(\zeta\omega_n) \Rightarrow \omega_n = \frac{4}{\zeta T_s}$$

In this way, the desired closed-loop polynomial will meet some design time requirements and will be expressed as

$$t_4 s^4 + t_3 s^3 + t_2 s^2 + t_1 s + t_0 \quad (24)$$

With known parameters $t_i > 0$.

Remark 1: There is no loss in generality if the polynomial (24) is chosen to be monic.

The characteristic equation of the control system in Fig. 2 corresponds to the expression $1 + G_1(s)G_p(s) = 0$. To represent the plant $G_p(s)$, the approximation through the first-order Padé model given by (6) will be used, and for the controller $G_1(s)$, equation (19) will be employed. The characteristic polynomial results in

$$\begin{aligned} &d_2 s^4 + (a_1 d_2 + d_1 - b_1 n_2) s^3 + (a_0 d_2 + \\ &a_1 d_1 + b_0 n_2 - b_1 n_1) s^2 + (a_0 d_1 + b_0 n_1 - \\ &b_1 n_0) s + b_0 n_0 \end{aligned} \quad (25)$$

To consider the approximation of (4) by the Taylor series expansion given by (5), it is enough to set $b_1 = 0$ in equation (25). In the following, a result is presented that allows calculating the constants of the controller $G_1(s)$ in the form (19).

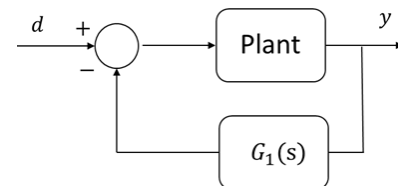


Figure 2. Block diagram for computing the controller $G_1(s)$.

Theorem 2: The constants d_2, d_1, e_2, e_1 , and e_0 of the controller $G_1(s)$ in equation (19) are given by solving the set of simultaneous equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a_1 & 1 & -b_1 & 0 & 0 \\ a_0 & a_1 & b_0 & -b_1 & 0 \\ 0 & a_0 & 0 & b_0 & -b_1 \\ 0 & 0 & 0 & 0 & b_0 \end{bmatrix} \begin{bmatrix} d_2 \\ d_1 \\ e_2 \\ e_1 \\ e_0 \end{bmatrix} = \begin{bmatrix} t_4 \\ t_3 \\ t_2 \\ t_1 \\ t_0 \end{bmatrix} \quad (26)$$

Proof: Straightforward, by equating the coefficients of the polynomials (24) and (25).

A legitimate question is why equation (11) of the PID controller is represented in form (19). If equation (11) is used directly to construct the characteristic polynomial in (25), the resulting system of equations is nonlinear and overdetermined instead of the simple form obtained in (26).

Remark 2: The determinant of the coefficient matrix in (26) is equal to $b_0(b_0^2 + a_1b_0b_1 + a_0b_1^2)$. For it to be zero, $b_0 = 0$ must occur. But this is impossible since the gain K in (4) must also be zero. That is, (26) has a unique solution, so one can always find values of the controller parameters K_{p1} , K_{i1} , K_{d1} , and τ_D using (16) from Theorem 1.

The following result shows that the controller $G_1(s)$, with the constants d_2 , d_1 , e_2 , e_1 , and e_0 calculated by solving (26), ensures that the component of the controlled output $y(t)$ in response to a constant disturbance signal $d(t)$ tends asymptotically to zero.

Theorem 3: If in the control system of Fig. 2, the controller parameters in (19) are selected by solving the set of simultaneous equations (26) then the component of the disturbance response at the controlled output $y(t)$ approaches zero asymptotically with a desired overshoot (OS) and settling time (T_s) when a step-type disturbance signal $d(t)$ is applied.

Proof: Given the specifications OS and T_s , and using the dominant pole guarantee criterion, the desired closed-loop polynomial (24) is obtained. For a disturbance signal of constant but unknown amplitude ρ , we have $D(s) = \rho/s$. Using (2), we can compute the steady-state value of the response component of the controlled output as

$$\lim_{s \rightarrow 0} s \frac{G_{yd}(s)D(s)}{Y(s)}$$

Employing (6) and (19) in the expression for G_{yd} , we obtain

$$\lim_{s \rightarrow 0} \frac{s^2(-b_1s + b_0)(d_2s + d_1)}{(t_4s^4 + t_3s^3 + t_2s^2 + t_1s + t_0)s} \frac{\rho}{s}$$

Since the limit is equal to zero, the theorem is established.

Remark 3: It should be noted that if the Taylor approximation (5) is used for the FOPTD plant, Theorem 3 remains valid.

3.2 Synthesis of controller $G_{12}(s)$

To synthesize the $G_{12}(s)$ controller, a criterion for assigning zeros to the closed-loop transfer function (3) will be used. For ease of reference, equation (3) is rewritten as

$$G_{yr}(s) = \frac{Y(s)}{R(s)} = \frac{G_{12}(s)G_p(s)}{1 + G_1(s)G_p(s)} \quad (27)$$

Using in (27), equations (6), (19), and (21) to represent $G_p(s)$, $G_1(s)$, and $G_{12}(s)$, respectively, results in

$$G_{yr}(s) = \frac{-b_1n_2s^3 + (b_0n_2 - b_1n_1)s^2 + (b_0n_1 - b_1n_0)s + b_0n_0}{t_4s^4 + t_3s^3 + t_2s^2 + t_1s + t_0} \quad (28)$$

A zero assignment criterion selects the G_{12} controller constants in (21) by making.

$$\begin{aligned} b_0n_0 &= t_0 \\ b_0n_1 - b_1n_0 &= t_1 \\ b_0n_2 - b_1n_1 &= t_2 \end{aligned}$$

Or equivalent,

$$\begin{aligned} n_0 &= t_0/b_0 \\ n_1 &= \frac{t_1 + b_1n_0}{b_0} \\ n_2 &= \frac{t_2 + b_1n_1}{b_0} \end{aligned} \quad (29)$$

Theorem 4: If the controller parameters in (21) are calculated using (29), then the controlled output $y(t)$ in the control system of Fig. 1 will track with zero steady-state error a reference signal of step, ramp, or parabola type.

Proof: Using (28) and taking the Laplace transform of equation (1) gives

$$E(s) = [1 - G_{yr}(s)]R(s)$$

The steady-state value of the tracking error is obtained by

$$e(\infty) = \lim_{s \rightarrow 0} s[1 - G_{yr}(s)]R(s)$$

Or, equivalently, through the expression

$$e(\infty) = \lim_{s \rightarrow 0} s \left[\frac{N(s)}{t_4s^4 + t_3s^3 + t_2s^2 + t_1s + t_0} \right] R(s)$$

Where

$$N(s) = t_4s^4 + (t_3 + b_1n_2)s^3 + (t_2 - b_0n_2 + b_1n_1)s^2 + (t_1 - b_0n_1 + b_1n_0)s + (t_0 - b_0n_0)$$

Considering $R(s) = \mu/s^k$ with μ a constant value, and $k = 1, 2$, or 3 depending on whether the reference signal is a step, a ramp, or a parabola, respectively. Using (29), the above limit tends to zero; thus, the theorem is established.

3.3 Synthesis of controller $G_2(s)$

The synthesis of this controller is immediate. Using (21) gives $G_2 = G_{12} - G_1$, and the PID controller constants in (20) are computed by

$$\begin{aligned} f_0 &= n_0 - e_0 \\ f_1 &= n_1 - e_1 \\ f_2 &= n_2 - e_2 \end{aligned} \quad (30)$$

It is recalled again that the constants of the PID controllers in the form of equations (11), (12) and (14) can be obtained from the constants computed in the form of equations (19), (20) and (21), respectively, using equation (16) of Theorem 1.

3.4 Synthesis procedure for the 2-DoF PID controller

For ease of reference, Table 1 summarizes the design procedure for controllers $G_1(s)$, $G_{12}(s)$, and $G_2(s)$.

Table 1. Design procedure for the 2-DoF PID controller.

Input	Constants b_1, b_0, a_1 , and a_0 in equations (5) or (6), depending on the time-delay θ_d approximation for the FOPTD model of the plant. Constants t_0, t_1, t_2, t_3 , and t_4 of the desired closed-loop polynomial in (24) from the OS and T_s design specifications.
Step 1	Solve (26) to compute constants d_2, d_1, e_2, e_1 , and e_0 for controller $G_1(s)$ in (19).
Step 2	Apply (16) to find controller constants K_{p1}, K_{i1}, K_{d1} , and τ_d in (11) ($\tau_d = \tau_{d1} = \tau_{d2}$).
Step 3	Use (29) to determine the constants n_2, n_1 , and n_0 for controller $G_{12}(s)$ in (21).
Step 4	Use (30) to determine the constants f_2, f_1 , and f_0 for controller $G_2(s)$ in (20).
Step 5	Apply (16) to find constants K_{p2}, K_{i2}, K_{d2} , and τ_d in (12).
Output	Controllers $G_1(s)$ and $G_2(s)$ in (11) and (12), respectively.

4 Results

In this section, several examples highlight different aspects of the design method for FOPTD systems (examples 1 - 3) and how the technique can be applied to other types of systems (examples 4 and 5).

4.1 Example 1

A heat flow process (Teppa-Garran et al., 2025a) is considered, which consists of a fiberglass duct with a heater and a blower located at one end and three temperature sensors along the duct. The controlled output corresponds to the temperature, and the control signal is the voltage applied to the heating element (the blower voltage is kept constant). The transfer function gives the temperature model at the third (furthest) sensor.

$$\frac{Y(s)}{U(s)} = \frac{6.1}{28s + 1} e^{-0.85s} \quad (31)$$

Using equations (5) and (6) to approximate the time delay in (31) results in the following design plant models, respectively.

$$G_p(s) = \frac{0.256}{s^2 + 1.212s + 0.042} \quad (32)$$

$$G_p(s) = \frac{-0.218s + 0.513}{s^2 + 2.389s + 0.084} \quad (33)$$

To compute the desired closed-loop polynomial in (24), the design specifications were $OS = 10\%$ and $T_s = 40$ s. The two fast poles are repeated and fixed ten times, the real part of the dominant poles. Applying the design procedure specified in Table 1, the controllers (11) and (12) parameters are computed, and their values are given in Table 2. It can be seen that controller G_2 is PD. To generate the following figures, the FOPTD model (31) represents the plant in the control system of Fig. 1. For controllers G_1 and G_2 , the values of Table 2 are employed depending on the time-delay approximation method. Fig. 3 shows the response to a step-type disturbance. Figures 4, 5, and 6 display the tracking of a step, ramp, and parabola input reference, respectively. It can be seen that the disturbance rejection is achieved in the desired settling time, but not with the desired overshoot. The fact that an irrational model, very different from the linear one used in the design of the controllers, is employed to represent the plant in the simulation produces differences. Tracking is attained for all the reference inputs, being best for the parabola and slightly deteriorating until the step. Fig. 7 illustrates the tracking of a reference input composed of a combination of ramps and steps, and Fig. 8 presents the evolution of the control signal for this case. The tracking is very satisfactory for both approximations of the delay-time, using a lower control effort for the Padé approximation.

Table 2. Constant values of PID controllers in Equations (11) and (12) for Example 1.

	K_{p1}	K_{i1}	K_{d1}	K_{p2}	K_{i2}	K_{d2}	τ_{d1} τ_{d2}
Padé	1.39	0.14	0.42	0.16	0	0.98	4.84
Taylor	0.68	0.06	0.0005	0.16	0	4.67	1.01

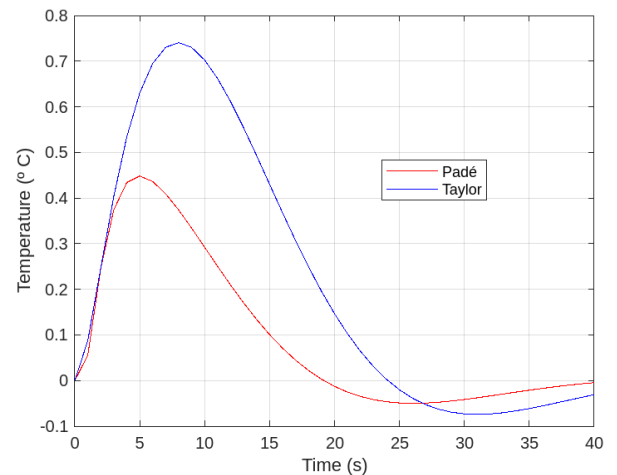


Figure 3. Step disturbance response for Example 1.

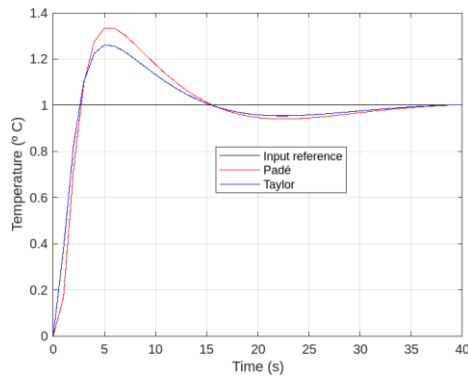


Figure 4. Step tracking response for Example 1.

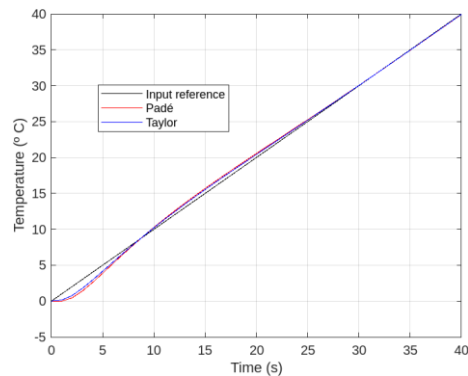


Figure 5. Ramp tracking response for Example 1.

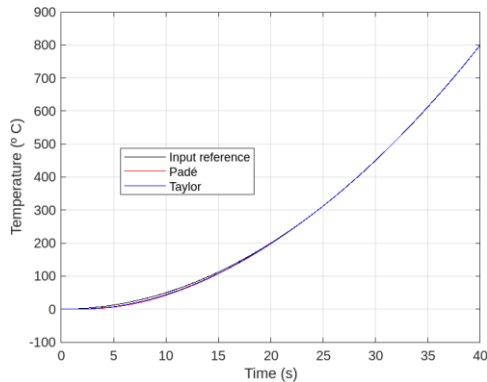


Figure 6. Parabola tracking response for Example 1.

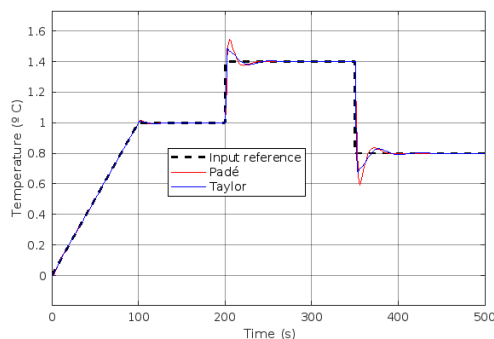


Figure 7. Temperature tracking response for the heat flow process comparing the Padé and Taylor approximations.

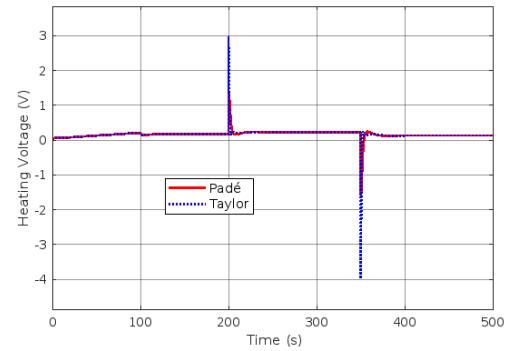


Figure 8. The heating voltage applied for the heat flow process.

4.2 Example 2

In the previous example, the relationship $\theta_d/T < 1$ was fulfilled in (4). In this example, we will deal with the more demanding situation where $\theta_d/T > 1$. For this purpose, it is considered a high-order process described as

$$\frac{Y(s)}{U(s)} = \frac{1}{(s+1)^{10}} \quad (34)$$

Employing the least-squares fitting between process and model frequency responses (Hang and Bi, 1997), the following FOPTD model is obtained.

$$\frac{Y(s)}{U(s)} = \frac{1}{2.72s+1} e^{-7.69s} \quad (35)$$

By applying the design procedure of Table 1, the parameters of the controllers (11) and (12) are given in Table 3. To compute the desired closed-loop polynomial in (24), the design specifications were $OS = 10\%$ and $T_s = 80$ s. The two fast poles are repeated and fixed five times the real part of the dominant poles.

Figure 9 shows the tracking of a combined reference input and the rejection of a constant disturbance signal applied from time $t = 400$ s. The FOPTD model of the plant (35) was used for the simulation. It can be seen that the performance using the Taylor or Padé approximations for the delay time is similar.

Fig. 10 compares the performance of the 2-DoF controller designed with the proposed method (Table 3, Taylor) with that of a 1-DoF PID tuned by the popular Chien-Hrones-Reswick (CHR) method (Teppa-Garran et al., 2021). It may seem that the latter's performance is superior, but it should be noted that the simulation for this case uses the linear model obtained from (35) by employing equation (5). When the FOPTD model (35) is used directly to represent the plant, the PID-CHR controller fails to stabilize the plant, as illustrated in Fig. 11.

Table 3. Constant values of PID controllers in Equations (11) and (12) for Example 2.

	K_{p1}	K_{i1}	K_{d1}	K_{p2}	K_{i2}	K_{d2}	τ_{d1} τ_{d2}
Padé	0.490	0.096	0.698	1	0	0.508	20.51
Taylor	0.116	0.091	0.322	1	0	1.065	9.77

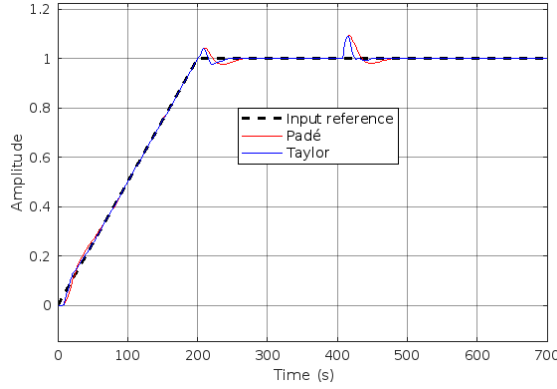


Figure 9. Tracking and disturbance-step rejection for Example 2 by comparing the Padé and the Taylor approximations.

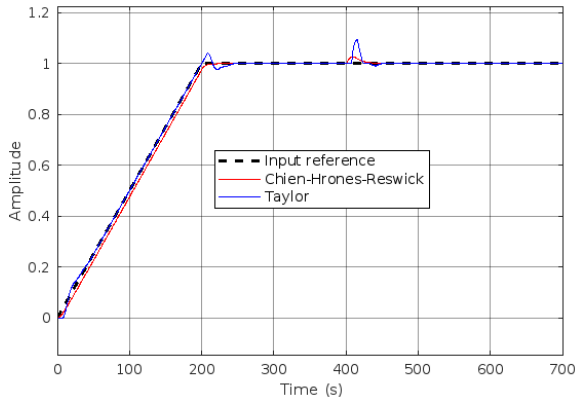


Figure 10. Tracking and disturbance-step rejection for Example 2, comparing the proposed method with a PID tuned by the CHR method.

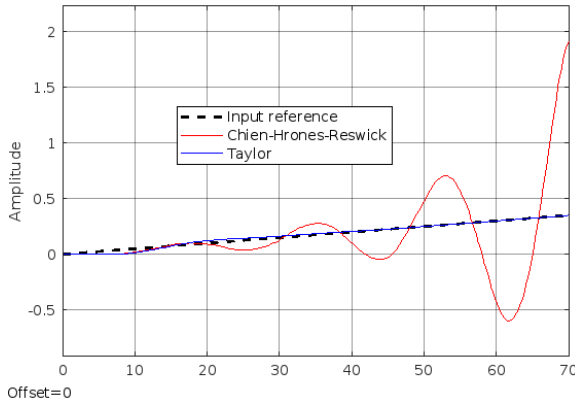


Figure 11. The PID controller tuned by the CHR method fails to stabilize the system in Example 2.

4.3 Example 3

Now we consider an open-loop unstable system described by the following FOPTD model (Yuce, 2023).

$$\frac{Y(s)}{U(s)} = \frac{1}{s-1} e^{-0.4s} \quad (36)$$

The desired closed-loop polynomial is computed for the specifications $OS = 0.05\%$ and $T_s = 20$ s. The two fast poles are repeated and fixed ten times, the real part of the dominant poles. Using the Taylor method (5) to approximate the time delay in (36) and applying the design procedure of Table 1 gives the controllers.

$$G_1(s) = 1.251 + \frac{0.046}{s} + \frac{0.283s}{0.345s + 1} \quad (37)$$

$$G_2(s) = -1 + \frac{1.74s}{0.345s + 1} \quad (38)$$

Using equations (36) – (38), the control system of Fig. 1 gives the response of Fig. 12. It can be appreciated that the correct tracking of the trapezoidal reference input and the rejection of a constant disturbance of negative amplitude applied from time $t = 25$ s.

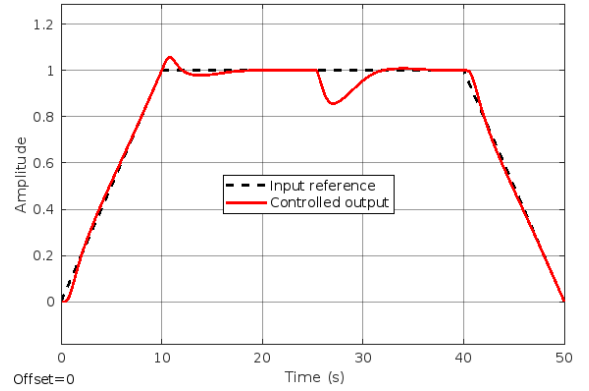


Figure 12. Tracking and disturbance rejection for an open-loop unstable process in Example 3.

4.4 Example 4

This example shows that the proposed method can be adapted to plants other than the FOPTD system. The state equations for a nonlinear model of a coupled tank system (Teppa-Garran et al., 2025b) are

$$\begin{aligned} \dot{x}_1(t) &= -0.904\sqrt{x_1(t)} + 0.258u(t) \\ \dot{x}_2(t) &= 0.904\sqrt{x_1(t)} - 0.508\sqrt{x_2(t)} \\ y(t) &= x_2(t) \end{aligned} \quad (39)$$

The variables x_1 and x_2 are the levels of tanks 1 and 2, respectively. They are restricted to the interval $[0, 30]$ cm. The control signal $u(t)$ is the voltage applied to a pump, limited to the range $[0, 21]$ V, and the controlled output $y(t)$ is the second tank level. To use the design method, the equation (39) is linearized at the point $(15, 15)$ cm, resulting in the transfer function.

$$G_p(s) = \frac{0.0302}{s^2 + 0.183s + 0.0077} \quad (40)$$

Equation (40) has the form (5) (Taylor). Hence, a 2-DoF PID can be designed using the proposed method. To that end, the desired closed-loop polynomial is computed for the

specifications $OS = 0.05\%$ and $T_s = 50$ s. The two fast poles are repeated and fixed ten times, the real part of the dominant poles. The design procedure of Table 1 gives the controllers.

$$G_1(s) = 2.232 + \frac{0.181}{s} + \frac{18.071s}{0.634s + 1} \quad (41)$$

$$G_2(s) = 0.255 + \frac{9.556s}{0.634s + 1} \quad (42)$$

Figure 13 shows the tracking response of the second-level tank to an input reference combining different ramps and steps. From the time instant $t = 325$ s, a constant disturbance signal is applied. The nonlinear model of the tank given in (39) has been used for the simulation. Figure 14 shows the voltage of the pump; it can be seen that it is always within the limits of pump operation.

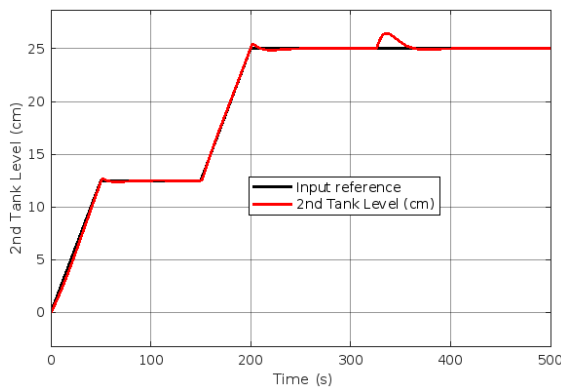


Figure 13. The second tank level response of the coupled tank system is shown in Example 4.

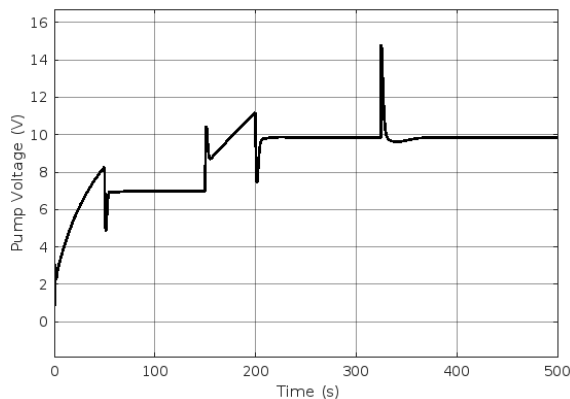


Figure 14. The voltage applied to the pump of the coupled tank system in Example 4.

4.5 Example 5

Consider a large chemical plant with the following transfer function (Tewari, 2003).

$$\frac{Y(s)}{U(s)} = \frac{0.0033}{s^3 + 0.630s^2 + 0.109s + 0.0033} \quad (43)$$

The output is the temperature, and the input is the mass flow rate of the Xylene gas. To obtain the design model in the form (5), the order reduction method (Kuo, 1991) is applied, resulting in

$$G_p(s) = \frac{0.0078}{s^2 + 0.242s + 0.0078} \quad (44)$$

The desired closed-loop polynomial is computed for the specifications $OS = 0.1\%$ and $T_s = 30$ s. The two fast poles are repeated and fixed ten times, the real part of the dominant poles. The design procedure of Table 1 gives the controllers.

$$G_1(s) = 26.445 + \frac{4.308}{s} + \frac{214.66s}{0.372s + 1} \quad (45)$$

$$G_2(s) = 1 + \frac{83.5s}{0.372s + 1} \quad (46)$$

The satisfactory temperature tracking of the large chemical plant can be seen in Fig. 15. The third-order model of the plant (43) is used in the simulation. The complete rejection of a constant disturbance signal applied from time $t = 200$ s can also be observed.

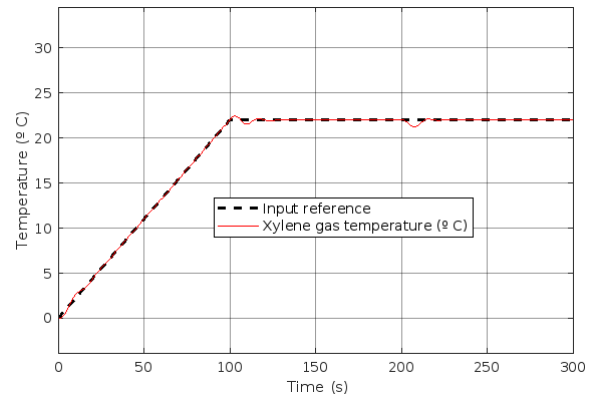


Figure 15. Temperature tracking and disturbance rejection for the large chemical plant of Example 5.

Discussion and Conclusions

A simple model-based method for the design of two-degree-of-freedom PID controllers based on a polynomial approach for FOPTD systems is proposed in this work. The problems of constant disturbance signal rejection and tracking step, ramp, or parabola reference inputs are solved independently. The constant disturbance rejection problem is solved by imposing a dominant pole guarantee criterion that allows choosing a desired overshoot and settling time in the temporal response. The tracking problem is solved by assigning zeros to the closed-loop transfer function.

Three examples of FOPTD system models are considered to cover the main cases that may arise in practical situations. That is, when the ratio of the delay time to the system time constant is greater than or less than one, and when the system is open-loop unstable. No specific advantages were found in using the Taylor or Padé methods to approximate the delay time in the FOPTD system. Other

two examples show that although the method was designed for FOPTD systems, it can be extended to different systems.

A weakness of the method is that the assignment of zeros imposed to solve the tracking problem can deteriorate the overshoot and settling time conditions of the transient component of the response. This is because the zeros cannot be fixed arbitrarily; their values come from solving equations that depend primarily on the characteristic polynomial coefficients, which rely on the overshoot and settling time specifications. What we wish to point out is that specifications could generate dominant zeros (near the imaginary axis of the complex plane), which cause a further increase in overshoot.

Another contribution of this work is that the proposed method allows tuning the four parameters of a PID controller expressed in the parallel form given by equation (9). In one example, it is observed that current tuning procedures, which initially focus on tuning the proportional, integral, and derivative constants using well-known rules and then adjusting the derivation filter parameter through predefined relationships, can result in poor performance.

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Pedro Teppa-Garrán received the B.S. degree in Electrical Engineering in 1990 at the Universidad Metropolitana (UNIMET) in Caracas, Venezuela, the Master degree in Electronic Engineering in 1994 and the Master degree in Mathematics in 1998; both of them, from the Universidad Simón Bolívar (USB), Caracas, Venezuela. He also received the Ph.D. in Control Systems in 2003 at Université Paul Sabatier in Toulouse, France and completed a Postdoctoral Research in LAAS-CNRS during 2012-2013 in Toulouse, France. He is currently a full professor of UNIMET
 <https://orcid.org/0000-0001-6384-3185>

Luis Caraballo: Electrical Engineer from UNIMET.
 c.luis@correo.unimet.edu.ve
 <https://orcid.org/0009-0004-5507-272X>

Germain Garcia received his diploma in engineering from the Institut National des Sciences Appliquées (INSA), Toulouse, France, in 1984. He also received his Ph.D. degree in Automatic Control from the INSA in 1988 and the Habilitation à Diriger des Recherches (HDR) in 1997 from the same university. He is currently working at the Laboratoire d'Architecture des Systèmes of the Centre National pour la Recherche Scientifique (LAAS-CNRS), Toulouse, France as full professor of INSA.
garcia@laas.fr

 <https://orcid.org/0000-0002-7147-5105>