A review of Max-Plus Algebra approach and some extensions to discrete event systems

Una revisión del Álgebra Max-Plus y algunas extensiones a sistemas de eventos discretos

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Abstract

The basic form of the systems we will study is $x_i(k+1) = max\{a_{i1} + x_1(k), a_{i2} + x_2(k), \dots, a_{in} + x_n(k)\} = \max_j\{a_{ij} + x_j(k)\}$, $i = 1,2,\dots,n$. *It is common in practice to change the notation somewhat. Addition* $+$ *will be written as* $\otimes$ *and max will be written as* $\oplus$. *This change of notation makes the resemblance with conventional linear difference systems visible:* $x_i(k+1) = \oplus_j \{a_{ij} \otimes x_j(k)\}$, $i = 1, 2, \dots, n$, *which in vector notation will be written as* $x(k+1) = A \otimes x(k)$. *Of the latter equation one speaks as a linear (difference) equation in the max-plus algebra, this in clear analogy with linear difference equations in the conventional, 'plus-times', algebra. We will briefly mention the following specializations and/or extensions: axiomatic foundations, minimal realizations, stochastic discrete event systems, min-max-plus systems and nonexpansive mappings, numerical procedures, 'continuous' discrete event systems and the Fenchel transform.*

Keywords: discrete event systems, max-plus algebra, γ −transform, extensions, the Fenchel transform.

Resumen

La forma básica de los sistemas que estudiaremos es $x_i(k+1) = max\{a_{i1} + x_1(k), a_{i2} + x_2(k), \dots, a_{in} + x_n(k)\} = \max_j\{a_{ij} + x_j(k)\}, i = 1,2,\dots,n$. *En la práctica es común cambiar la notación. Además, + será denotada por* $\otimes$, *y max será escrito como* $\oplus$.*Este cambio de notación se hace en términos de los sistemas de diferencia lineales convencionales visibles:* $x_i(k+1) = \oplus_j \{a_{ij} \otimes x_j(k)\}$, $i = 1, 2, \dots, n$, *en los cuales, la notación vectorial se escribirá como* $x(k+1) = A \otimes x(k)$. *En la última ecuación uno se refiere a una ecuación (en diferencias) lineal en el algebra max-plus, lo cual es una clara analogía con las ecuaciones lineales en diferencias en el algebra 'max-plus'. Mencionaremos brevemente las siguientes especializaciones y/o extensiones: fundamentaciones axiomáticas, realizaciones minimales, sistemas de eventos discretos estocásticos, sistemas min-max-plus y mapeos no expansivos, procedimientos numéricos, sistemas de eventos discretos 'continuos' y la transformada de Fenchel.*

Palabras clave: sistemas de eventos discretos, algebra max-plus, γ −transformada, extensions, transformada de Fenchel.

1 Introduction

The discrete event systems (DES) is a research area of current vitality. The development of this is largely stimulated by discovering general principles which are useful to a wide range of application domains. In particular, manufacturing systems, communication networks, and logistic systems. There are two key features that characterize these systems:

- Their dynamics are event-driven, i.e., the behavior of a DES is governed only by occurrences of different types of events overtimes rather than by ticks of a clock;

- At least some of the natural variables required to describe a DES are discrete (see Branicky, 1995). The theory of DES encompasses a variety of classes of problems and of modelling approaches (see Murata, 1989). In this paper, one will discover the max-plus algebra approach (Coen et al., 1999), in which, in addition to the precise ordering of the events, the timing of the events plays an essential role (see Anderson, 2002).

From a formal point of view, a DES can be thought of as a dynamical system (see Baccelli et al., 2001), with a state space and a state-transition mechanism. The event-driven nature of a DES forces us to seek new mathematical frameworks for modelling and analysis (see Komenda et al., 2018), since differential or differential equations no longer provide an adequate setting.

Finally, we include some extensions; namely, axiomatic foundations, minimal realizations, stochastic discrete event systems, min-max-plus systems (see De Schutter and Van Den Boom, 2008) and nonexpansive mapping, numerical procedures, 'continuous' discrete event systems and the Fenchel transform.

2 Motivation and Petri net

It is assumed that the reader is familiar with the basic properties of Petri net. It will be shown that the max-plus algebra is extremely suitable in describing the timed behavior of tokens in so-called event graphs, which form a subclass of Petri nets. In order to set the notation and the stage, we do start with some formal definitions.

A Petri net is a pair $(\boldsymbol{G}, \boldsymbol{b})$, where $\boldsymbol{G} = (\boldsymbol{E}, \boldsymbol{V})$ is a bipartite graph with a finite number of nodes ($\boldsymbol{V}$) which are partitioned into the disjoint sets $\boldsymbol{P}$ and $\boldsymbol{Q}$; $\boldsymbol{E}$ consist of pairs of the form $(\boldsymbol{P_i}, \boldsymbol{q_j})$, and $(\boldsymbol{q_j}, \boldsymbol{P_i})$ with $\boldsymbol{P_i} \in \boldsymbol{P}$ and $\boldsymbol{q_j} \in \boldsymbol{Q}$. The initial marking $\boldsymbol{b}$ is an m-vector, with m being the number of elements in P, of nonnegative integers. The elements of P are called places, those of Q are called transitions. The number of elements in these sets are m and n respectively. The elements of the vector b denote the number of tokens in the respective places. One talks about a timed Petri net if time durations are associated with places and transitions.

A (timed) Petri net is called a (timed) event graph if each place has exactly one upstream and one downstream transition.

A (timed) Petri net is called a (timed) state graph if each transition has exactly one upstream and one downstream place.

Note that the definitions of event graph and state graph are dual one to the other. In these definitions it was tacitly assumed that the networks are 'closed', i.e., all places (transitions) do have an upstream and a downstream transition (place). The definitions can be extended in the obvious way to include imput transitions (places), so-called sources, which do not have upstream places (transitions) and output transitions (places), so-called sinks, which do not have downstream places (transitions).

A transition can fire (or can start firing if there is a positive firing time) if all its (directly) upstream places contain at least one token (which must be 'enabled'). After the firing these tokens are removed and one token is added to each of the (directly) downstream places.

The firing time of a transition is the time that elapses between the starting and the completion of the firing of the transition. The holding time of a place is the time a token must spend in the place before it can contribute to the enabling of the downstream transitions.

An event graph with both firing times and holding times is equivalent to an event graph with only holding times (i.e., the firing time are zero). This equivalence means that the time instants at which the transitions fire are the same in both event graphs.

The number of tokens in any circuit of an event graph is constant. (*)

From now on we will only consider event graphs with firing times which are zero. Each place connects precisely one transition with precisely one (possibly different) transition. One says in such a situation that the upstream transition, say $\boldsymbol{q_j}$, is a predecessor of the downstream transition, say $\boldsymbol{q_i}$. Equivalently one can say that $\boldsymbol{q_i}$ is a successor of $\boldsymbol{q_j}$. One writes in such a case $\boldsymbol{j} \in \boldsymbol{\pi}^{-}(\boldsymbol{i})$ and $\boldsymbol{i} \in \boldsymbol{\pi}^{+}(\boldsymbol{j})$.

We make the explicit assumption that if a place connects two transitions such as just has been described, there is no other place with does exactly the same. In general, event graphs there can be more 'parallel' places in between two transitions of which one is the successor of the other. The reason for this restriction is purely a notational issue. The theory to be given can handle the more general situation routinely. We also make the assumption that the underlying network is strongly connected.

If a place exists between the transition $\boldsymbol{q_j}$ and $\boldsymbol{q_i}$ and $\boldsymbol{q_j}$ is upstream with regard to this place and $\boldsymbol{q_i}$ downstream, then the holding time of this place is indicated by $\boldsymbol{a_{ij}}$. The holding times are nonnegative real numbers. The number of tokens in this place is indicated by $\boldsymbol{b_{ij}}$. In the course of time, $\boldsymbol{b_{ij}}$ may change of course, but what is meant here, and also in the formulas to come,

$$\boldsymbol{\tau_i(x) = \max_{j\in\pi^-(i)}\{a_{ij} + \tau_j(x - b_{ij})\},\ i = 1, 2, \dots, n},$$

where $\boldsymbol{\tau_i(x)}$ denotes the earliest time instant at which transition $\boldsymbol{q_i}$ has fired $\boldsymbol{x}$ times, and

$$\boldsymbol{x_i(t) = \min_{j\in\pi^-(i)}\{b_{ij} + x_j(t - a_{ij})\}, i = 1, 2, \dots, n}, \qquad (**)$$

where $\boldsymbol{x_i(t)}$ denotes the number of firings of transition $\boldsymbol{q_i, i = 1, 2, \dots, n}$, which have taken place up to, and including, time t.

Rather than having used the conventional notation x for the state, we now used the symbol $\boldsymbol{\tau}$. The equations above describe the same underlying system and one equation is called the dual of the other. Note that x and $\boldsymbol{x_i}$ are integer-valued. The functions $\boldsymbol{x_i(t)}$ and $\boldsymbol{\tau_i(x)}$ are each other inverse in a way.

If in the original event graph there would have been a positive firing time, then the equations above do not exclude the possibility that a transition 'works' simultaneously on two or more tokens. If one wants to exclude this, a loop, including one place with one token, around the transition concerned should be added. The holding time of this new place is defined to be equal to the original firing time of this transition. This loop now takes care of the fact that in the equivalent event graph with only zero firing times, the transition cannot work on two or more tokens simultaneously anymore.

In a metropolitan area there are two railway stations, S_1 and S_2, which are interconnected by a railway system as indicated in Figure 1. This railway system consists of an inner circle and of two outer circles. The trains on these outer circles deliver and pick up passengers in the suburbs. The stations in the suburbs have not been drawn since they do not play any role in the model to be formulated.

Suppose there are four trains (two at each station) and the leave the stations at time 0, one along each track. They reach the other (or the same) station after a certain time which is indicated in the figure. The arriving trains at a station have to wait for each other such as to allow the passengers to change trains.

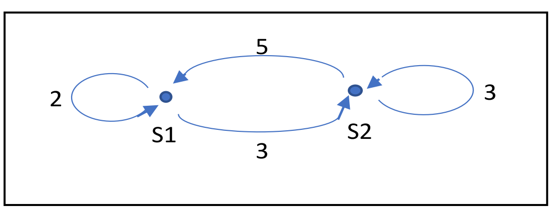

Fig. 1. The two stations example

Figure 1 can easily be redrawn as an event graph. The two stations are transitions and in the four railway tracks one can put a place. If a train is running along a track, one puts a token in the place corresponding to this track.

Suppose that there is not time table and that the trains leave directly after the change over of the passengers at the stations and that the time needed for change overs has been incorporated in the travelling time. This 'travelling time' was called 'holding time' in the theory of Petri nets. If this process of departing and arriving trains is continued, the departure time $\boldsymbol{x_i(k+1)}$ for the $\boldsymbol{(k+1)}$ −st departure at station $\boldsymbol{S_i}$ satisfies:

$$\boldsymbol{\begin{aligned} x_1(k+1) &= \max\{x_1(k) + 2, x_2(k) + 5\} \\ x_2(k+1) &= \max\{x_1(k) + 3, x_2(k) + 3\} \end{aligned}} \qquad \boldsymbol{(1)}$$

for k = 0, 1, 2, …., with $\boldsymbol{x_1 = 0, x_2 = 0}$, the evolution of equations (1) becomes

$$\boldsymbol{\underset{x(0)}{\begin{pmatrix}0\\0\end{pmatrix}} \to \underset{x(1)}{\begin{pmatrix}5\\3\end{pmatrix}} \to \underset{x(2)}{\begin{pmatrix}8\\8\end{pmatrix}} \to \underset{x(3)}{\begin{pmatrix}13\\11\end{pmatrix}} \to \underset{x(4)}{\begin{pmatrix}16\\16\end{pmatrix}} \to \cdots}$$

This pattern of departure times shows a periodic solution superimposed on a linear drift, the 'period' equals 2 and the average time between two subsequent departures is 4. From a time's table point of view, it is better to start with the initial departures $\boldsymbol{x_1 = 1, x_2 = 0}$, since then the evolution becomes

$$\boldsymbol{\underset{x(0)}{\begin{pmatrix}1\\0\end{pmatrix}} \to \underset{x(1)}{\begin{pmatrix}5\\4\end{pmatrix}} \to \underset{x(2)}{\begin{pmatrix}9\\8\end{pmatrix}} \to \underset{x(3)}{\begin{pmatrix}13\\12\end{pmatrix}} \to \cdots}$$

where the interdeparture time is now exactly 4 at each station and thus the departure times are very regular (they have period 1). By trial and error, it turns out that, whatever the initial condition, after possibly a short transient period of time, a periodic behavior of either period 1 or 2 is obtained with (average) interdeparture times 4. A solution with an (average) departure time smaller than 4 is not possible, since for a train to go around in the inner circle costs 3+5=8 times units. There are two trains on the inner circle and therefore the (average) interdeparture time is limited from below by 8/2=4.

With the above sketched railway system and trains, it is not possible to design a time table with interdeparture times smaller than 4. If one wants a faster time table, one must change the problem. To this end, let us add a train on the inner circle, such that three trains will run along this circle all the time. Suppose that initially this extra train is situated at station $\boldsymbol{S_1}$. The equations for the departure times now become

$$\boldsymbol{x_1(k+1) = \max\{x_1(k) + 2, x_2(k) + 5\}}, \qquad \boldsymbol{(2)}$$
$$\boldsymbol{x_2(k+1) = \max\{x_1(k-1) + 3, x_2(k) + 3\}}, \qquad \boldsymbol{(3)}$$

which can be rewritten as a set of first order equations as

$$x_1(k+1) = \max\{x_1(k)+2, x_2(k)+5\},$$
$$x_2(k+1) = \max\{x_3(k)+3, x_2(k)+3\} \quad (4)$$
$$x_3(k+1) = x_1(k)$$

with the initial condition $x_1 = 0, x_2 = 0, x_3 = 0$. The evolution of the latter set of equations becomes

$$\begin{pmatrix}0\\0\\0\end{pmatrix} \to \begin{pmatrix}5\\3\\0\end{pmatrix} \to \begin{pmatrix}8\\6\\5\end{pmatrix} \to \begin{pmatrix}11\\9\\8\end{pmatrix} \to \begin{pmatrix}14\\12\\11\end{pmatrix} \to \cdots$$
$$x(0) \quad x(1) \quad x(2) \quad x(3) \quad x(4)$$

which shows, after a transient part, a regular behavior of 'period' 1, with interdeparture times 3. This interdepartures time is caused by the outer loop at station S_2; on this loop there is one train which needs 3 times units to travel around. The inner loop is not the bottleneck anymore this inner loop itself would lead to a lower limit of 8/3 (travelling time of the loop divided by the number of trains on this loop). In order to lower the interdeparture times even more, one should add the next extra train to the outer loop of $\mathbf{S_2}$. If we do so, the equations for the departure times become

$$x_1(k+1) = \max\{x_1(k)+2, x_2(k)+5\},$$
$$x_2(k+1) = \max\{x_3(k)+3, x_2(k-1)+3\} \quad (5)$$
$$x_3(k+1) = x_1(k)$$

which can be rewritten as a set of first order equations as

$$x_1(k+1) = \max\{x_1(k)+2, x_2(k)+5\},$$
$$x_2(k+1) = \max\{x_3(k)+3, x_4(k)+3\}$$
$$x_3(k+1) = x_1(k) \quad (6)$$
$$x_4(k+1) = x_2(k)$$

If we start again with zero initial conditions, the solution become

$$\ldots \to \begin{pmatrix}10\\8\\8\\3\end{pmatrix} \to \begin{pmatrix}13\\11\\10\\8\end{pmatrix} \to \begin{pmatrix}16\\13\\13\\11\end{pmatrix} \to \begin{pmatrix}18\\16\\16\\13\end{pmatrix} \to \begin{pmatrix}21\\19\\18\\16\end{pmatrix} \to .. \quad (7)$$
$$x(3) \quad x(4) \quad x(5) \quad x(6) \quad x(7)$$

This solution has a 'period' 3 ($x_i(k+3) = x(k) + 8, k \geq 4$) and the average interdeparture time is 8/3, which is caused by the inner circle. Another solution results, with the same interdeparture time but with period 1, if one start with the initial condition $x_1(0) = 5, x_2(0) = \frac{8}{3}, x_3(0) = \frac{7}{3}, x_4(0) = 0$. One then has $x_i(k+1) = x_i(k) + \frac{8}{3}, i = 1,2,3,4$, and k = 0, 1, 2, ….

3 Structure

Consider the systems

$$x_i(k+1) = \max_j\{a_{i1} + x_1(k), a_{i2} + x_2(k), \ldots, a_{in} + x_n(k)\}$$
$$= \max_j\{a_{ij} + x_j(k)\}, i = 1,2,\ldots,n \quad (8)$$

Equivalent,

$$x_i(k+1) = \oplus\{a_{ij} \otimes x_j(k)\}, i = 1,2,\ldots,n, \quad (9)$$

with addition $+$ be written as $\otimes$ and $\mathbf{max}$ be written as $\oplus$, which in vector notation will be written as

$$x(k+1) = A \otimes x(k). \quad (10)$$

If it is clear from the context that the underlying algebra is the max-plus one, one even writes $x(k+1) = Ax(k)$, for (10). If the initial condition for (10) is $x(0) = x_0$, then

$$x(1) = A \otimes x_0,$$
$$x(2) = A \otimes x(1) = A \otimes (A \otimes x_0) = (A \otimes A) \otimes x_0$$
$$= A^2 \otimes x_0.$$

It can be shown that indeed $A \otimes (A \otimes x_0) = (A \otimes A) \otimes x_0$. For the example given in Section 2, it is easy to check this by hand. Instead of $A \otimes A$, we simply write A^2. We get, for the general case,

$$x(k) = \Big(\underbrace{A \otimes A \otimes \ldots \otimes A}_{k\ times}\Big) \otimes x_0 = A^k \otimes x_0.$$

The matrices $A^2, A^3, \ldots$ can be calculated directly. Let us consider the $A-$ matrix of (1), $A = \begin{pmatrix}2 & 5\\3 & 3\end{pmatrix}$, then

$$A^2 = \begin{pmatrix}\max\{2+2, 5+3\} & \max\{2+5, 5+3\}\\ \max\{3+2, 3+3\} & \max\{3+5, 3+3\}\end{pmatrix}$$
$$= \begin{pmatrix}8 & 8\\6 & 8\end{pmatrix}.$$

In general,

$$(A^2)_{ij} = \oplus_l\ a_{il} \otimes a_{lj} = \max_l\{a_{il} + a_{lj}\}. \quad (11)$$

In terms of the railway example, the quantity $(A^2)_{ij}$ can be interpreted as the máximum (with respect to l) of all connections from station S_j , via station S_l to station S_i. One speaks of paths of length two between the stations S_j and S_i. In graph-theory terminology, the stations are called nodes and the tracks between stations are called arcs. More generally, $(A^k)_{ij}$ denotes the maximum of all paths of length k, starting at node j and ending at node i.

In many networks such as a railway net there will not be an arc from each node to each other node. If there is not arc from node $\boldsymbol{S_j}$ to node $\boldsymbol{S_i}$ then the behavior of node $\boldsymbol{S_i}$ is not directly influenced by that of node $\boldsymbol{S_j}$. In such situation it is useful to consider the element $\boldsymbol{a_{ij}}$ to be equal to $-\infty$. In (8) a term $-\infty + \boldsymbol{x_j(k)}$ does not directly influence $\boldsymbol{x_i(k+1)}$ as long as $\boldsymbol{x_j(k)}$ is finite. The number $-\infty$ will occur frequently in the sequel and it will be indicated by $\boldsymbol{\varepsilon}$.

Linear systems in the max-plus algebra with inputs and outputs are given by

$$\begin{cases} x(k+1) = Ax(k) \oplus Bu(k) \\ y(k) = Cx(k) \end{cases} \quad (12)$$

which is short-hand notation for

$$x_i(k+1) = \max\{a_{i1} + x_1(k), \dots, a_{in} + x_n(k), b_{i1} + u_1(k), \dots, b_{im} + u_m(k)\}, i = 1, 2, \dots, n;$$
$$y_i(k) = \max\{c_{i1} + x_1(k), \dots, c_{in} + x_n(k)\}, i = 1, 2, \dots, p.$$

A seeming generalization of (10) is

$$x(k+1) = A_0 x(k+1) \oplus A_1 x(k) \oplus \dots \oplus A_{l+1} x(k-1), \quad (13)$$

which is implicit in $\boldsymbol{x(k+1)}$ and which has extra delays. By repeated substitution of the whole right-hand side of (13) for the term $\boldsymbol{x(k+1)}$ in this right-hand side, one gets

$$x(k+1) = A_0^* A_1 x(k) \oplus \dots \oplus A_0^* A_{l+1} x(k-1) \quad (14)$$

where $\boldsymbol{A_0^* = I \oplus A_0 \oplus A_0^2 \oplus A_0^3}$ .... The notation I refers to the identity matrix in the max-plus algebra: it has zeros on the main diagonal and $\boldsymbol{\varepsilon}$'s elsewhere. Equation (14) only makes sense if $\boldsymbol{A_0^*}$ is well defined (its elements are finite or $\boldsymbol{\varepsilon}$). This is for instance the case if the precedence graph of $\boldsymbol{A}$ does not contain circuits, because then $\boldsymbol{A^k = \varepsilon}$ for $\boldsymbol{k \geq n}$. Equation (14) can be rewritten as a first order difference equation by augmenting the state space. This is a standard trick in system theory and has already been used in section 2.

4 Behavior

Given a square matrix $\boldsymbol{A}$, we consider the problema of existence of eigenvalues and eigenvectors in the max-plus algebra, that is, the existence of $\boldsymbol{\lambda}$ and $\boldsymbol{v \neq \varepsilon}$ such that

$$Av = \lambda v. \quad (15)$$

This equation has to be interpreted in the max-plus algebra sense; the expresión $\boldsymbol{\lambda v}$ means that one adds $\boldsymbol{\lambda}$ to each component of $\boldsymbol{v}$. We already have seem examples of eigenvalues and eigenvectors in section 2; $\boldsymbol{v}$ correspond to an initial state resulting in a solution with 'period' 1 and $\boldsymbol{\lambda}$ is the interdeparture time.

Before formulating a theorem about eigenvalues, some graph theory must be recapitulated. In the following definition the starting point is a square matrix, the entries of which may again assume the 'value' $\boldsymbol{\varepsilon}$.

Definition 1 (Precedence graph). The precedence graph of an $\boldsymbol{n \times n}$ matrix $\boldsymbol{A}$ is a weighted digraph with $\boldsymbol{n}$ nodes and an arc $\boldsymbol{(j, i)}$ if $\boldsymbol{a_{ij} \neq \varepsilon}$, in which case the weight of this arc receives the numerical value of $\boldsymbol{a_{ij}}$. The precedence graph is denoted by $\boldsymbol{G(A)}$.

It is not difficult to see that any weighted digraph $\boldsymbol{G = (V, E)}$, with $\boldsymbol{V}$ being the set of nodes and $\boldsymbol{E}$ being the set of arcs, is the precedence graph of an appropriately defined square matrix. The weight $\boldsymbol{a_{ij}}$ of the arc from node $\boldsymbol{j}$ to node $\boldsymbol{i}$ is defined as the $\boldsymbol{ij - th}$ entry of a matrix $\boldsymbol{A}$. If an arc does not exist, the corresponding entry of A becomes $\boldsymbol{\varepsilon}$. The matrix A thus defined has $\boldsymbol{G}$ as its precedence graph.

As we have seen before, the element $\boldsymbol{(i, j)}$ of $\boldsymbol{A^k = A \otimes A \otimes \dots A \otimes A}$, considered within the max-plus algebra denotes the maximum weight with respect to all paths of length k, which go from node j to node i. If no such path exists, then $\boldsymbol{(A^k)_{ij} = \varepsilon}$. The weight of a path $\boldsymbol{\rho}$ is denoted $\boldsymbol{|\rho|_w}$ and its length is denoted $\boldsymbol{|\rho|_l}$.

Definition 2. The mean weight of a path is defined as the sum of the weights of the individual arcs of this path, divided by the length of this path. If the path is denoted by $\boldsymbol{\rho}$, then the mean weight equals $\frac{|\rho|_w}{|\rho|_l}$. If such a path is a circuit one talks about the mean weight of the circuit, or simply the cycle mean.

We are interested in the maximum of these cycle means, where the maximum is taken over all circuits in the graph. This number will be called the maximum cycle mean. If the cycle mean of a circuit equals the maximum cycle mean, then the circuit is called critical. The graph consisting of all critical circuits (if there happen to be more than one) is called the critical graph and denoted by $\boldsymbol{G^c}$. In the following theorem the notion 'strongly connected' digraph is used. A graph is called strongly connected if there exist a path from any node to any other node. The matrix corresponding to a strongly connected graph is called irreducible.

Theorem 1. We are given a square matrix A. If $\boldsymbol{G(A)}$ is strongly connected, then there exist one an only one eigenvalue and at least one eigenvector. The eigenvalue is equal to the maximum cycle mean of the graph: $\boldsymbol{\lambda = \max_{\chi} \frac{|\chi|_w}{|\chi|_l}}$, where $\boldsymbol{\chi}$ ranges over the set of circuits of $\boldsymbol{G(A)}$.

Definition 3 (Cyclicity of a graph). Given a strongly connected graph, its cyclicity equals the greatest common divisor of the lengths of all its circuits. The cyclicity of an arbitrary graph (which may consist of several strongly connected subgraphs) equals the least common multiple of the cyclicities of all its maximal strongly connected subgraphs.

Definition 4 (Cyclicity of a matrix). A matrix $\boldsymbol{A}$ is said to be cyclic, if there exist scalars $\boldsymbol{M}, \boldsymbol{\lambda}$ and d, such that for all $\boldsymbol{m} \geq \boldsymbol{M}, \boldsymbol{A}^{\boldsymbol{m}+\boldsymbol{d}} = \boldsymbol{\lambda}^{\boldsymbol{d}}\boldsymbol{A}^{\boldsymbol{m}}$. The least such d is called the cyclicity of A. The quantity $\boldsymbol{\lambda}$ equals the maximun cycle mean of A.

The expression $\boldsymbol{A}^{\boldsymbol{m}+\boldsymbol{d}} = \boldsymbol{\lambda}^{\boldsymbol{d}}\boldsymbol{A}^{\boldsymbol{m}}$ in the definition above must be interpreted in the max-plus algebra sense of course. Thus $\boldsymbol{\lambda}^{\boldsymbol{d}}$ in the max-plus algebra means $\boldsymbol{d\lambda}$ in the conventional algebra and $\boldsymbol{\lambda}^{\boldsymbol{d}}\boldsymbol{A}^{\boldsymbol{m}}$ refers to the addition of $\boldsymbol{\lambda}^{\boldsymbol{d}}$ to each element of $\boldsymbol{A}^{\boldsymbol{m}}$**.**

Theorem 2. Any irreducible matrix is cyclic. The cyclicity of the irreducible matrix A equals the cyclicity of $\boldsymbol{G}^{\boldsymbol{c}}(\boldsymbol{A})$, being the critical graph corresponding to matrix A.

Example 1. Consider the A-matrix of (6):

$$\boldsymbol{A} = \begin{pmatrix} 2 & 5 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 3 & 3 \\ 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix}.$$

The corresponding precedence graph has three circuits, viz from node 1 to node 1 with cycle mean 2/1=2; from node 1 to 3 to 2 to 1 with cycle mean (0+3+5)/3=8/3; from node 2 to 4 to 2 with cycle mean (0+3)/2=3/2 (it is tacitly assumed here that node i corresponds to $\boldsymbol{x_i}$). The maximum cycle means equals 8/3. There is only one critical circuit. The cyclicity of the critical graph (which equal the critical circuit) equal 3. The quantities of definition 4 are M=5, $\boldsymbol{\lambda} = \boldsymbol{8/3}$ and d=3;

$$\boldsymbol{A}^{\boldsymbol{8}} = \begin{pmatrix} 20 & 23 & 24 & 24 \\ 19 & 20 & 21 & 21 \\ 18 & 21 & 20 & 20 \\ 15 & 18 & 19 & 19 \end{pmatrix} = \left(\frac{8}{3}\right)^{3} \boldsymbol{A}^{5}$$
$$= 8\begin{pmatrix} 12 & 15 & 16 & 16 \\ 11 & 12 & 13 & 13 \\ 10 & 13 & 12 & 12 \\ 7 & 10 & 11 & 11 \end{pmatrix}.$$

5 The $\boldsymbol{\gamma}$ −Transform

Conventional linear systems with inputs and outputs are of the form (12), though (12) itself has the max-plus algebra interpretation. This equation, now considered in the conventional way, is a representation of a linear system in the time domain. Its representation in the Z-domain equals

$$\boldsymbol{Y}(\boldsymbol{Z}) = \boldsymbol{C}(\boldsymbol{ZI} - \boldsymbol{A})^{-1}\boldsymbol{BU}(\boldsymbol{Z}),$$

where $\boldsymbol{Y}(\boldsymbol{Z}), \boldsymbol{U}(\boldsymbol{Z})$ are defined by

$$\boldsymbol{Y}(\boldsymbol{Z}) = \sum_{i=0}^{\infty} \boldsymbol{y}(\boldsymbol{i})\boldsymbol{Z}^{-1},\ \boldsymbol{U}(\boldsymbol{Z}) = \sum_{i=0}^{\infty} \boldsymbol{U}(\boldsymbol{i})\boldsymbol{Z}^{-1},$$

where it is tacitly assumed that the system was at rest for $\boldsymbol{t} \leq \boldsymbol{0}$ and where I refer to the unit matrix in the conventional algebra. The matrix $\boldsymbol{H}(\boldsymbol{Z}) = \boldsymbol{C}\,(\boldsymbol{ZI} - \boldsymbol{A})^{-1}\boldsymbol{B}$ is called the transfer matrix of the system.

In the max-plus algebra context, the Z-transform also exists, but here it is customary to refer to it as the $\boldsymbol{\gamma}$ −transform where $\boldsymbol{\gamma}$ operates as $\boldsymbol{Z}^{-1}$ and is assumed to be real-valued. For instance, the $\boldsymbol{\gamma}$ −transform of u is defined as

$$\boldsymbol{U}(\boldsymbol{\gamma}) = \bigoplus_{i=0}^{\infty} \boldsymbol{u}(\boldsymbol{i}) \otimes \boldsymbol{\gamma}^{\boldsymbol{i}} \qquad (16)$$

and $\boldsymbol{Y}(\boldsymbol{\gamma})$ and $\boldsymbol{X}(\boldsymbol{\gamma})$ are defined likewise. Multiplication of (12) by $\boldsymbol{\gamma}^{\boldsymbol{k}}$ yields

$$\begin{cases} \boldsymbol{\gamma}^{-1}\boldsymbol{x}(\boldsymbol{k}+\boldsymbol{1})\boldsymbol{\gamma}^{\boldsymbol{k}+1} = \boldsymbol{A} \otimes \boldsymbol{x}(\boldsymbol{k})\boldsymbol{\gamma}^{\boldsymbol{k}} \oplus \boldsymbol{B} \otimes \boldsymbol{u}(\boldsymbol{k})\boldsymbol{\gamma}^{\boldsymbol{k}} \\ \boldsymbol{y}(\boldsymbol{k})\boldsymbol{\gamma}^{\boldsymbol{k}} = \boldsymbol{C} \otimes \boldsymbol{x}(\boldsymbol{k})\boldsymbol{\gamma}^{\boldsymbol{k}} \end{cases} \qquad (17)$$

If these equations are summed with respect to $\boldsymbol{k} = \boldsymbol{0}, \boldsymbol{1}, \ldots$, and if we add $\boldsymbol{\gamma}^{-1}\boldsymbol{x_0}$ to both sides of the first equation thus obtained, then we obtain

$$\begin{cases} \boldsymbol{\gamma}^{-1}\boldsymbol{X}(\boldsymbol{\gamma}) = \boldsymbol{A} \otimes \boldsymbol{X}(\boldsymbol{\gamma}) \oplus \boldsymbol{B} \otimes \boldsymbol{U}(\boldsymbol{\gamma}) \oplus \boldsymbol{\gamma}^{-1}\boldsymbol{x_0} \\ \boldsymbol{Y}(\boldsymbol{\gamma}) = \boldsymbol{C} \otimes \boldsymbol{X}(\boldsymbol{\gamma}). \end{cases} \qquad (18)$$

The first of these equations can be solved be first multiplying (max-plus algebra), equivalently adding (conventional), left and right-hand side by $\boldsymbol{\gamma}$ and then repeatedly substituting the right-hand side for $\boldsymbol{X}(\boldsymbol{\gamma})$ within this right-hand side. This results in

$$\boldsymbol{X}(\boldsymbol{\gamma}) = (\boldsymbol{\gamma A})^{*}(\boldsymbol{\gamma BU}(\boldsymbol{\gamma}) \oplus \boldsymbol{x_0}).$$

Thus, we obtain $\boldsymbol{Y}(\boldsymbol{\gamma}) = \boldsymbol{H}(\boldsymbol{\gamma})\boldsymbol{U}(\boldsymbol{\gamma})$, provided that $\boldsymbol{x_0} = \boldsymbol{\varepsilon}$, and where the transfer matrix $\boldsymbol{H}(\boldsymbol{\gamma})$ is defined by

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{\gamma}) &= \boldsymbol{C} \otimes (\boldsymbol{\gamma A})^{*} \otimes \boldsymbol{\gamma} \otimes \boldsymbol{B} \\ &= \boldsymbol{\gamma CB} \oplus \boldsymbol{\gamma}^{2}\boldsymbol{CAB} \oplus \boldsymbol{\gamma}^{3}\boldsymbol{CA}^{2}\boldsymbol{B} \oplus \ldots \end{aligned} \qquad (19)$$

The expression $\boldsymbol{Y}(\boldsymbol{\gamma}) = \boldsymbol{H}(\boldsymbol{\gamma})\boldsymbol{U}(\boldsymbol{\gamma})$ is the max-plus algebra equivalent of $\boldsymbol{Y}(\boldsymbol{Z}) = \boldsymbol{H}(\boldsymbol{Z})\boldsymbol{U}(\boldsymbol{Z})$ in the conventional system theory. If one writes

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{z}) = \boldsymbol{C}(\boldsymbol{zI} - \boldsymbol{A})^{-1}\boldsymbol{B} &= \boldsymbol{C}\left(\frac{\boldsymbol{1}}{\boldsymbol{\gamma}}\boldsymbol{I} - \boldsymbol{A}\right)^{-1}\boldsymbol{B} \\ &= \boldsymbol{\gamma C}(\boldsymbol{I} - \boldsymbol{\gamma A})^{-1}\boldsymbol{B} \\ &= \boldsymbol{\gamma C}(\boldsymbol{I} + \boldsymbol{\gamma A} + \boldsymbol{\gamma}^{2}\boldsymbol{A}^{2} + \cdots)\boldsymbol{B}, \end{aligned}$$

one has obtained the equivalence of (19) in the conventional sense.

The transfer matrix (19) is defined by means of an infinite series and the convergence depends on the value of $\boldsymbol{\gamma}$**.** If the series is convergent for $\boldsymbol{\gamma = \gamma'}$, then it is also convergent for all $\boldsymbol{\gamma' s}$ which are smaller than $\boldsymbol{\gamma'}$**.** If the series does not converge, it still has a meaning as a formal series.

Exactly as in conventional system theory, the transfer matrix is especially useful when subsystems are combined to build larger systems, by means of parallel, series and feedback connections. For instance, the product of two transfer matrices (of which it is tacitly assumed that the sizes of these matrices are such that the multiplication is possible), is a new transfer matrix which refers to a system which consists of the original systems put into a series connection.

Suppose that $\boldsymbol{H(\gamma)}$ is a scalar function, i.e., the system has one input and one output. The term $\boldsymbol{\gamma^k C A^{k-1} B}$ in (19) can be written in conventional algebra as $\boldsymbol{C_{k-1} + k\gamma}$ (where $\boldsymbol{C_{k-1}}$ represents the coefficient $\boldsymbol{CA^{k-1}B}$) which is a straight line with slope k. The transfer function can be viewed as the maximum (of an infinite number) of such lines and hence is a continuous, piecewise linear and convex function of the variable $\boldsymbol{\gamma}$**.**

6 Some Extensions

6.1 Axiomatic foundations

The operations $\oplus$ and $\otimes$ defined on the set R can also be defined with respect to a more general set of elements D. One then speaks of a dioid (sometimes also referred to as a semiring).

Definition 5. A dioid is a set D endowed with two operations denoted $\oplus$ and $\otimes$ (called 'sum' or 'addition', and 'product' or 'multiplication') obeying the following axioms:

1. For all $a, b, c \in D, (a \oplus b) \oplus c = a \oplus (b \oplus c)$;
2. For all $a, b \in D, a \oplus b = b \oplus a$;
3. For all $a, b, c \in D, (a \otimes b) \otimes c = a \otimes (b \otimes c)$;
4. For all $a, b, c \in D$, $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c), c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$.

This is right, respectively left, distributivity of multiplication jwith respect to addition. One statement does not follow from the other since multiplication is not assumed to be commutative.

5. Exist $\varepsilon \in D: \forall a \in D, a \oplus \varepsilon = a$;
6. $\forall a \in D, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$;
7. Exist $e \in D: \forall a \in D, a \otimes e = e \otimes a = a$;
8. $\forall a \in D, a \oplus a = a$.

Definition 6. A dioid is commutative if multiplication is commutative.

With the noticeable exception of axiom 8, most the axioms of dioids are required for rings too. Indeed, axiom 8 is the most distinguishing feature of dioids. Because of this axiom, addition can not be cancellative, that is, $\boldsymbol{a \oplus b = a \oplus c}$ does not imply b=c in general. Multiplication is not necessarily cancellative either (of course, because of axiom 6, cancellation would anyway only apply to elements different from $\boldsymbol{\varepsilon}$). For an example in which multiplication is not cancellative take $\boldsymbol{D} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and define $\oplus$ as max and $\otimes$ as min.

It is easily shown that in dioids the distributivity with respect to matrices also holds, i.e., $\boldsymbol{A \otimes (B \otimes C) = (A \otimes B) \otimes C}$, where these multiplilcations only make sense if the matrices have appropriate dimensions.

6.2 Minimal realizations

In Section 5 it was shown how to derive the transfer matrix of a system if the representation of the system in the 'event domain' is given. This event domain representation is characterized by the matrices A, B and C. Now one could pose the opposite question; how to obtain an event domain representation, or equivalently, how to find A, B and C if the transfer matrix is given. In the conventional linear system theory, the corresponding theory is known as the realization theory and one speaks of a minimal realization if the sizes of A, B and C are as small as possible.

The simplest formulation of the (minimal) realization problem in the max-plus algebra is probably as follows. Let G be a sequence of real numbers $\{\boldsymbol{g_l}\}_{\boldsymbol{l=0}}^{\infty}$ and let $\boldsymbol{A} \in \boldsymbol{R^{n\times m}}, \boldsymbol{x_0} \in \boldsymbol{R^{n\times 1}}, \boldsymbol{C} \in \boldsymbol{R^{1\times n}}$ be such that $\boldsymbol{g_j = C \otimes A^j \otimes x_0, j = 0, 1, \ldots}$, then G is said to be reproduced by the discrete event system $\boldsymbol{x(k+1) = A \otimes x(k), x(0) = x_0}$ and $\boldsymbol{y(k) = C \otimes x(k)}$. Given a sequence produced in this way, finds its realization of the smallest dimension. This realization problem has attracted a lot of attention recently, but for the moment it remains unclear whether an exact algorithmic procedure of polynomial complexity can be found for the general case.

In most approaches the theorem of Cayley-Hamilton, suitably adapted to the max-plus algebra plays a crucial role.

6.3 Stochastic Discrete Event Systems

The evolution equation studied in this part is

$$\boldsymbol{x(k+1) = A(k) \otimes x(k), \qquad k = 0, 1, 2, \ldots \qquad (20)}$$

with some initial condition $\boldsymbol{x(0)}$**.** Some (or all) entries of $\boldsymbol{A(k)}$ are stochastic. We assume that:

- The underlying distribution functions do not depend on k.
- The stochastic entries can assume only a finite number of different values. It will also be assumed that

these values are finite, though the method to be described can be generalized to the case that $-\infty$ is also allowed as a value.

- A(k) and A(n) are independent stochastic matrices for $k \neq n$ (extensions exist for problems where A(k) and A(k+1) are correlated).
- No correlation between stochastic entries of exists, though such correlations can be treated rather routinely.
- G(A(k)) is strongly connected (if this assumption is true for one k, it automatically is true for all k due to the second assumption above).

The quantity of central interest is

$$\lim_{k\to\infty} E\left(\frac{x_i(k)}{k}\right), \qquad (21)$$

For an arbitrary i, being the average cycle time for component i. This quantity is a kind of 'average cycle time'; it can be proved that this average cycle time is independent of i.

6.4 Min-max-plus systems and nonexpansive mappings

Referring to the right-hand side of (8), one can define a max-plus expression as a (finite) set of $x_i + a_{ij}$ terms, connected by the max operator. Similarly, one defines a min-max-plus expression as a (finite) set of $x_i + a_{ij}$ term, connected by both the max and min operators. An example of such an expression is

$$\max\{x_1 + 7, \min\{x_2 - 4, x_3 + 1, \max\{x_1, x_4 + 2\}\}\}.$$

With respect to the operator, we introduced the neutral element $-\infty$. Since we now also deal with the min operator, it is convenient to introduce its neutral element $+\infty$ also. Exactly as one can define a max-plus system by means of max-plus expressions, as (8) can be viewed, one can define a min-max-plus system. Such systems are nonlinear in the max-plus algebra (because of the presence of the min operator) and also nonlinear in the min-plus algebra (because of the presence of the max operator). Not with standing this higher complexity, various results about min-max plus systems are known. Specifically necessary and sufficient conditions are known under which the evolution of (subclasses of) minx-max-plus systems show a regular pattern as the evolution of max-plus systems does.

Min-max-plus systems are quite naturally imbedded in the class of so-called nonexpansive mappings, which also are know to have the possibility of periodic behavior. For more information on such mappings, we end this subsection by the definition of nonexpansive mappings.

Definition 7. A mapping f, which maps R^n into R^n is called nonexpansive, if $\left\|f(z) - f(\xi)\right\| \le \left\|z - \xi\right\|$, (22) for arbitrary $z, \xi \in R^n$, and where the norm is an arbitrary $\|.\|_p$ norm, with $1 \le p \le \infty$.

6.5 Numerical procedures

In this subsection we will confine ourselves to numerical procedures which yield the eigenvalue and eigenvector of a matrix A, as expressed by (15).

Theorem 3. Given is an $n \times n$ matrix A, with corresponding precedence graph $G = (V, E)$. The maximum cycle mean is given by

$$\lambda = \max_{i=1,\dots,n} \min_{k=0,1,\dots,n-1} \frac{(A^n)_{ij} - (A^k)_{ij}}{n-k}, \forall j. \qquad (23)$$

In this equation, A^n and A^k are to be evaluated in the max-plus algebra; the other operations (subtraction and division) are conventional ones. This theorem is known as Karp's theorem. This theorem yields the eigenvalue but does not give information about the eigenvectors. For that purpose construct the matrix B by subtracting λ, obtained by Karp's theorem, from all elements of A. The máximum circuit weight of $G(B)$ equals 0. Hence $B^* = I \oplus B \oplus B^2 \oplus \dots$ and $B^+ = BB^*$ exist. Matrix B^+ has some columns with diagonal elements equal to zero. To prove this, pick a node k of a circuit x, such that $x \in \arg_\xi \frac{|\xi|_w}{|\xi|_l}$. The maximum weight of paths from node k to k is 0. Therefore $B_k^+ = 0$. Let B_k denote the k-th column of B. Then, since $B^+ = BB^*$ and $B^* = I \oplus B^+$ (I is the identity matrix), for that k, $B_k^+ = B_k^*$ imply $BB_k^* = B_k^+ = B_k^*$ imply $AB_k^* = \lambda B_k^*$. Hence, $V = B_k^+ = B_k^*$ is an eigenvector of A, corresponding to the eigenvalue λ.

A few other numerical approaches exist which calculate the eigenvalue and/or eigenvector:

- Study of the zero (s) of the characteristic equation in the max-plus algebra yields the eigenvalue.
- By means of linear programming techniques.
- Consider (7). Calculate $x(k), k = 0,1,\dots$, starting from an arbitrary initial condition, until a state becomes linearly dependent on a state already calculated ($x(7) = 8 \otimes x(4)$). Now 8/7-4 equals the eigenvalue and $\frac{(x(4)+x(5)+x(6))}{7} - 4$.

6.6 'Continuous' Discrete event systems and the Fenchel transform

The central equations of this part are (1) and (2). It is assumed now that in addition to t and τ_i, also x and x_i are real-valued, and so are the quantities b_{ij}. The interpretation of these equations is still a (strongly connected) network with n transitions (also called nodes now). These nodes can now fire continuously. The intensity of this firing is indicated by $v_i(t)$. Quantity $x_i(t)$ denotes again the total amount produced by node i up to (and including) time t. As initial condition it is assumed that $\boldsymbol{x_i(0)=0}$. The production of a continuously firing transition in sent with unit speed along the outgoing arcs to the downstream transitions. Thus along an arc there is a continuous flow. The intensity of this flow is $\boldsymbol{\varphi_i(t,l)}$, where l is the parameter indicating the exact location along the arc; l=0 coincides with the beginning of the arc, l= coincides with the end of the arc, where it is assumed that the downstream transition is $\boldsymbol{q_j}$. As long as the parameters lie in appropriate intervals, we have $\boldsymbol{\varphi_i(t,l)=\varphi_i(t+s,l+s)}$. Moreover, $\boldsymbol{\varphi_i(t,l)=\varphi_i(t-l,0)=v_i(t-l)}$.

At time t, the total amount of material along the arc from $\boldsymbol{q_i}$ to $\boldsymbol{q_j}$ equals

$$\int_{l=0}^{l=a_{ji}} \varphi_i(t,s)ds\,. \qquad (24)$$

The quantities $\boldsymbol{b_{ij}}$ satisfy $\boldsymbol{b_{ij}=\int_0^{a_{ij}}\varphi(0,l)dl}$. The integrand and the integral in (24) must be considered with some care. It is quite well possible that the integrand will contain $\boldsymbol{\delta}$ −functions. This will particularly happen at the end of an arc, when material must wait there to be processed by the downstream transition because the other incoming arcs to the same transition have brought in less material sofar. If $\boldsymbol{q_k}$ is a downstream transition to both $\boldsymbol{q_i}$ and $\boldsymbol{q_j}$ and if $\boldsymbol{x_i(t)<x_j(t)}$, then $\boldsymbol{\varphi_{kj}}$ will start to build a $\boldsymbol{\delta}$ −function at $\boldsymbol{l=a_{kj}}$, from t onwards. Of course this $\boldsymbol{\delta}$ −function can disappear again later on if $\boldsymbol{x_i(s)>x_j(s)}$ for an $\boldsymbol{s}$ −value with $\boldsymbol{s>t}$. The total amount of material along an arc, as expressed by (24), will in general be time dependent. Many standard results of Section 4 on periodic behavior remain valid for this continuous version of flow on networks.

Theorem 4. Along a circuit the total amount of material is constant. In formula, if the circuit x is characterized by the transition $\{\boldsymbol{q_{i1},q_{i2},\dots,q_{i(k+1)}=q_{i1}}\}$,
then

$$\sum_{l=1}^{k}\int_0^{a_{il+1,il}} \varphi_{il}(t,s)ds$$

is constant (it does not depend on time).

This is the ‘continuous event’ analogue of (*). Note that the total amount of material in the network is not necessarily constant.

Definition 8. Given a circuit

$$\tau=\{q_{i1},q_{i2},\dots,q_{i(k+1)}=q_{i1}\},$$

its cycle mean is defined as $\frac{|\tau|_w}{|\tau|_l}$, where the weight $\boldsymbol{|\tau|_w}$ and the length $\boldsymbol{|\tau|_l}$ (assumed to be positive) are defined as

$$|\tau|_w=\sum_{l=1,\dots,k} a_{i(l+1),i_l}\,,\qquad |\tau|_l=\sum_{i=1,\dots,k} b_{i(l+1),i_l}.$$

Definition 9. The circuits which have the maximum cycle mean are called critical. The corresponding cycle mean is indicated by $\boldsymbol{\lambda}$.

Theorem 5. Equations (**) has a solution $\boldsymbol{x_i(t)=\frac{1}{\lambda}t+d_i}$, with appropriately chosen constants $\boldsymbol{d_i}$.

Just as there are linear difference equations versus Z-transforms, and similarly, linear differential equations versus Laplace transforms, we have in the max-plus algebra setting their counterparts. For the discrete event systems we had the max-plus algebra systems versus the $\boldsymbol{\gamma}$ transform. Fort he continuous flow variation we have the description above versus a slight variation of the Fenchel transform, called Fenchel* transform. This Fenchel* transform turns out to be the max-plus algebra variant of Laplace transform. The Fenchel transform of a function is also referred to as the conjugate function. We conclude with its definition.

Definiton 10. The Fenchel transform $\boldsymbol{J(f)}$ of a function f in a Hilbert space $\boldsymbol{H}$ is a function in the dual space $\boldsymbol{H}^*$;

$$\forall c\in H^*:[J^*(f)](c)=\sup_{z\in H}\{\langle c,z\rangle+f(z)\}. \qquad (25)$$

Usually one confines the definition to convex functions f. Note the resemblance between (16) and (25).

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