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Generalized mutual exclusion: Petri nets for resource allocation with choice in manufacturing

Exclusión mutua generalizada: redes de Petri para la asignación de recursos con elección en manufactura

Rosales-Moret, José-René[1]*; Mata-Díaz, Guelvis[1]; Ferrer-Guillén, María[2] ; Mantilla Morales, Gisella[3]; Coello-Vera, Adalberto-Antonio[4]; Pérez-Pirela, Maribel[3]
[1] Faculty of Sciences, Department of Mathematics, University of Los Andes, Mérida, Venezuela.
[2] Faculty of Engineering, Department of Calculating, University of Los Andes, Mérida, Venezuela.
[3] Technical University of Manabí, Ecuador.
[4] Technical University of the State of Quevedo, Ecuador.
*joserene@ula.ve

Abstract

This paper presents a definition of generalized mutual exclusion, along with an application in a manufacturing context. This includes a methodology for developing the most relevant concepts in this class of systems. More precisely, our general problem is stated as follows: given the specifications or conditions in a Manufacturing System, where a shared resource is channeled to carry out operations with choice, to construct a Petri Net-based model containing the aforementioned exclusion, whose structure and initial marking guarantee boundedness, deadlock-freeness, and reversibility.

Keywords: discrete event systems; Petri nets; generalized mutual exclusion; manufacturing systems.

Resumen

En este trabajo se presenta una definición de la exclusión mutua generalizada, junto con una aplicación en un contexto de manufactura, esto incluye una metodología para el desarrollo de los conceptos más relevantes en dicha clase de sistemas. Más precisamente, nuestro problema general es establecido como sigue: dadas las especificaciones o condiciones en un Sistema de Manufactura, donde un recurso compartido es canalizado para llevar a cabo operaciones con elección, construir un modelo basado en Redes de Petri conteniendo una exclusión como la mencionada, cuya estructura y marcación inicial garanticen acotamiento, no bloqueo y reiniciabilidad.

Palabras clave: sistemas de eventos discretos; redes de Petri; exclusión mutua generalizada; sistemas de manufactura.

1 Introduction

The evolution toward flexible manufacturing and modern industry has exponentially increased the complexity of Discrete Event Systems (DES), as discussed in (Zhou, 1990). In these automated environments, the optimization of shared resources and conflict prevention are fundamental challenges (Li et al., 2012). When two or more independent processes compete for a shared and indivisible resource, the classic mutual exclusion problem arises; for a precise discussion on this, refer to (Mata et al., 2016).

In its conventional or simple form, mutual exclusion dictates that when a resource is used by multiple processes, it is allocated to one of them, used in a predefined sequence, and subsequently released. To illustrate this, consider an Articulated Robotic Crane shared by three independent processes in a manufacturing cell: Assembly, Inspection, and Cooling. Under the simple mutual exclusion scheme, if the crane serves the Assembly process, it executes a fixed and strict sequence of movements (pick up part, transfer, release) and then becomes free. The same occurs if it serves Inspection or Cooling; the start and end actions have no ambiguity and do not involve decision-making (they are modeled with a single pair of start and end transitions per process).

Modern industrial reality demands flexibility (Coello et al., 2022; De Souza & Loreiro, 2018). What happens if the system requires operations with choice? Suppose that, upon finishing the Inspection process, the crane must decide the destination of the part: if the part is acceptable, the crane transfers it to the Cooling phase; but if it has defects, it chooses a different route to deposit it at a rework station. Or, in the Loading stage, the crane could choose between different conveyor belts depending on their availability. This scenario of alternative routes and decision-making completely breaks the simple mutual exclusion paradigm, generating the unavoidable need to formulate a generalized mutual exclusion.

The objective of this paper is to establish a formal framework based on Petri Nets (PN) that supports this theoretical generalization and demonstrates its practical feasibility. The paper is organized as follows: Section 2 presents the basic definitions of PN and their qualitative properties; Section 3 details the modeling methodology; Section 4 formally defines simple mutual exclusion and generalized mutual exclusion, demonstrating the preservation of their qualitative properties; Section 5 presents an application that implements the proposed approach; and finally, Section 6 presents the conclusions.

2 Basic definitions of Petri Nets and qualitative properties

To unambiguously represent manufacturing systems, we rely on Petri Net theory. Mathematically, these nets are built upon sets that model the system's conditions (places) and the events that alter these conditions (transitions). The system's dynamics—that is, how its state evolves over time due to the occurrence of discrete events—are governed by a token distribution (marking) and strict enabling and firing rules. These fundamental concepts are formalized below.

Definition 1 (Petri Net). A Petri Net (PN) is a quadruple $R = (L, T, E, S)$, where $L = \{l_1, l_2, ..., l_n\}$ is a finite set of places, $T = \{t_1, t_2, ..., t_k\}$ is a finite set of transitions, $L \cap T = \emptyset$, $E : T \rightarrow L^\infty$ is an input function such that $E(t)$ is the multiset of input places for t; and $S : T \rightarrow L^\infty$ is an output function such that $S(t)$ is the multiset of output places for t.

Definition 2 (Marked Petri Net). A marked PN is a pair $M = (R, m_0)$, where R is a PN and $m_0: L \rightarrow \mathbb{N}$ is an initial marking function. For each $l_i \in L$, $m_0(l_i) \in \mathbb{N}$ determines the number of tokens in place l_i.

Observation 1. Marked PN can be represented by bipartite directed graphs, where places are represented by circles and transitions by bars. If l_i and l_j are entry and exit places respectively for a transition t, then there are $\#\ (l_i, E(t))$ directed arcs from the corresponding circle to the corresponding bar and $\#\ (l_j, S(t))$ directed arcs from the corresponding bar to the corresponding circle. Finally, the tokens are represented by points inside the circle and, consequently, the marking function is represented by the number of points at the entrances of the n-tuple.

Definition 3 (Enabled Transition). A transition $t \in T$ in a marked PN M is called enabled if $m(l_i) \geq \#\ (l_i, E(t))$ for all places $l_i \in L$. The set of enabled transitions at marking m is given by $\mathcal{E}(m) = \{t \in T: m(l_i) \geq \#\ (l_i, E(t)), \forall\, l_i \in L\}$.

Definition 4 (Marking Change Function). Given a marked PN, the partial marking change function $\delta: \mathbb{N}^{|L|} \times T \rightarrow \mathbb{N}^{|L|}$ is defined for (m, t) if and only if $t \in \mathcal{E}(m)$. The resulting marking $m' = \delta\ (m, t)$ is given by $m'(l_i) = m(l_i) - \#\ (li, E(t)) + \#\ (li, S(t))$.

Definition 5 (Reachability Set). Given a marked PN $M= (R, m_0)$, a marking m is reachable from m_0 if there exists a firing sequence σ such that $\delta\ (m_0, \sigma) = m$. The reachability set is defined as $A\ (R, m_0) = \{m \in \mathbb{N}^{|L|}: \exists\ \sigma, \delta\ (m_0, \sigma) = m\}$.

The validation of a model in engineering depends not only on correctly representing the sequence of operations, but also on ensuring that the physical system will operate without failures. This entails ensuring that physical resources are not overloaded (boundedness), that the system does not stall (deadlock-freeness), and that it can return to an idle state to initiate new cycles. To this end, the following qualitative properties are analyzed:

Definition 6 (Boundedness and Safeness). A place $l \in L$ is k-bounded if there exists $k \in \mathbb{N}$ such that $m(l) \leq k$ for all $m \in A\ (R, m_0)$. If all places are k-bounded, the net is k-bounded. In particular, if the net is **1**-bounded, it is called safe.

Definition 7 (Non-blocking). A marked PN M is called non-blocking if, for every marking $m \in A\ (R, m_0)$ and every transition $t_i \in T$, there exists a marking $m' \in A\ (R, m)$ such that $t_i \in \mathcal{E}(m')$.

Definition 8 (Reversibility). A marked PN M is called reversible if the initial marking $m_0 \in A\ (R, m)$ for every marking $m \in A\ (R, m_0)$.

3 Manufacturing Systems and Methodology

A manufacturing system can be described as a pair of sets: a set of activities and a set of resources. These interact to produce a product; thus, it is established that resources are utilized to carry out the pre-established activities within a system, according to a given schedule (admissible sequences).

Based on the aforementioned description, a methodology capable of establishing precedence relations in manufacturing systems is considered. To this end, a construction process based on the interpretation of places and transitions is initiated as follows:

1. The resources and activities required for the production of a product are established;

2. The activities are ordered hierarchically based on precedence relations, as dictated by the schedule;

3. For each activity, a place is created to represent its current state. Subsequently, an activity-start transition is created, and an arc from the transition to the place is included. Next, a transition corresponding to the activity's completion is created, including an arc from the place to the transition. Conventionally, the completion transitions of current activities serve as the start transitions for subsequent activities. A token present in a place corresponding to an activity indicates that said activity is being executed; conversely, the absence of tokens in the place indicates that it is not being executed;

4. For each activity, a place is created for every resource required to start the aforementioned activity. These places are connected to the activity's start transition via directed arcs from the places to the start transition. Subsequently, an arc is created from the activity's completion transition to any resource place that becomes available upon completion of the activity. If a place represents a resource condition, the number of tokens in the place indicates the availability of that resource; the absence of tokens indicates that the resource is unavailable;

5. An initial marking is established.

To formalize mutual exclusion, a classification of the places within a $\mathbf{PN}$ corresponding to manufacturing systems is assumed; that is, a partition of the set of places $\mathbf{L = L_o \cup L_f \cup L_v}$, where $\mathbf{L_o}$ denotes operations places, $\mathbf{L_f}$ denotes fixed-quantity resource places, and $\mathbf{L_v}$ denotes variable-quantity resource places.

Definition 9 (o-path). Let $\mathbf{y, x \in (L \cup T) \setminus L_o}$. An $\mathbf{o}$-path between $\mathbf{x}$ and $\mathbf{y}$ is any elementary path $\mathbf{x_1x_2...x_k}$ (sequence of nodes) such that $\mathbf{x_1 = x}$, $\mathbf{x_k = y}$, and $\mathbf{x_i \in L_o \cup T}$ for each $\mathbf{i}$ with $\mathbf{2 \leq i \leq k\text{-}1}$.

To model rigorously, the net must comply with two fundamental design conditions:

Definition 10 (Condition CL). Let $\mathbf{M = (R, m_0)}$ with $\mathbf{L = L_o \cup L_f \cup L_v}$ being a partition of $\mathbf{L}$. $\mathbf{M}$ has condition $\mathbf{CL}$ if, given $\mathbf{l' \in (L_f \cup L_v) \cap E(t)}$ such that $\mathbf{m_0(l') = 0}$, and for every $\mathbf{l \in L}$ with $\mathbf{l \neq l'}$, the following holds:

$$\mathbf{m_0(l)} = \begin{cases} \mathbf{0}, & \text{if} \quad \mathbf{l \in L_0} \\ \mathbf{\geq 1}, & \text{if} \quad \mathbf{l \in (L_f \cup L_v)} \end{cases}$$

$$\mathbf{\Rightarrow t \notin \mathcal{E}(m)}, \text{ for all } \mathbf{m \in A(R, m_0)}.$$

This condition ensures that a resource in $\mathbf{L_f \cup L_v}$ cannot be converted into a different resource. Any released resource must retain its nature.

Definition 11 (Condition S). A marked $\mathbf{PN}$ $\mathbf{M = (R, m_0)}$ with $\mathbf{R = (L_0 \cup L_f \cup L_v, T, E, S)}$ has condition $\mathbf{S}$ if for each pair $\mathbf{l, l' \in L_f \cup L_v}$ with $\mathbf{l \neq l'}$, and for every $\mathbf{t \in T}$ and $\mathbf{o}$-paths $\mathbf{c\ (l, t)}$ and $\mathbf{c\ (l', t)}$, if they exist, the first encounter between the $\mathbf{o}$-paths occurs at a transition.

This condition ensures that, if there are resource tokens from resources of different natures exist, their interaction (synchronization) must occur exclusively through a shared transition, never in a place.

4 Mutual Exclusion in Manufacturing and its Generalization

Mutual exclusion occurs when independent processes compete for a common indivisible resource. In the classical modeling of manufacturing systems, if a resource is shared by $\mathbf{k}$ distinct processes and each process executes a single routine with no alternatives, the use of said resource can be delimited by an exact pair of events. Formally, the $\mathbf{i}$-th process acquires the resource through a specific transition $\mathbf{t_{\alpha i}}$ and releases it through another transition $\mathbf{t_{\beta i}}$, while the shared resource is modeled by a place $\mathbf{l_E}$. This strict dynamics gives rise to the formal definition of simple mutual exclusion, from which important conservation properties for the net are derived.

Definition 12 (Simple k-mutual exclusion). Given $\mathbf{M = (R, m_0)}$, $\mathbf{R = (L_o \cup L_f \cup L_v, T, E, S)}$. A $\mathbf{k}$-mutual exclusion $\mathbf{(k\text{-}EM)}$ for $\mathbf{M}$ is a pair $\mathbf{(l_E, \varphi)}$ such that:

1. $\mathbf{l_E \in L_f, m_0(l_E) = 1}$ and $\mathbf{\varphi = \{(t_{\alpha 1}, t_{\beta 1}), ..., (t_{\alpha k}, t_{\beta k})\}}$, $\mathbf{k \geq 1}$, is a finite set of transition pairs that satisfy the following conditions:

   (a) If $\mathbf{i \neq j}$, $\mathbf{1 \leq i, j \leq k}$, then $\mathbf{t_{\alpha i} \neq t_{\beta j}}$, $\mathbf{t_{\alpha i} \neq t_{\alpha j}}$ and $\mathbf{t_{\beta i} \neq t_{\beta j}}$;

   (b) If $\mathbf{t_{\alpha i} \neq t_{\beta i}}$ y $\mathbf{1 \leq i \leq k}$, then $\mathbf{\#\ (l_E, E(t_{\alpha i})) = \#\ (l_E, S(t_{\beta i})) = 1}$ and $\mathbf{\#\ (l_E, S(t_{\alpha i})) = \#\ (l_E, E(t_{\beta i})) = 0}$. Furthermore, If $\mathbf{t} \notin \mathbf{T_\alpha \cup T_\beta}$ with $\mathbf{T_\alpha = \{t_{\alpha i} : 1 \leq}$

$i \le k\}$ and $T_\beta = \{t_{\beta i} : 1 \le i \le k\}$, then $\#\ (l_E, E(t)) = \#\ (l_E, S(t)) = 0$;

(c) If $1 \le i \le k$, $l \in L_v$ y $l \in c(t_{\alpha i})$, then $t_{\beta i} \in c(t_{\alpha i})$;

(d) If $1 \le i \le k$ and $c(t_{\alpha i}) \cap (L_f \cup L_f) = \{l_E\}$, then $t_{\beta i} \in c(t_{\alpha i})$;

(e) If $1 \le i \le k$, $t \in T$ y $t \in c\ (t_{\alpha i}, t_{\beta i})$, then $t \in c\ (t_{\alpha i}, t_{\beta i}) = o$-path.

2. If $L_f \cup L_v$ satisfy the conditions **CL** and **S**, $t \in T_\beta$ and $t' \in T_\alpha$, then $\nexists\ c\ (t, t') = o$-path;

3. There exists a marking m_1 and a two-stop sequence of transitions σ that contains no transitions in T_α such that $t \in \mathcal{E}\ (\delta\ (m_1, \sigma))$, for all $t \in T_\alpha$.

4. For every marking m_1, if $t_{\alpha j}$ is fired at $m \in A(R, m_1)$ then for every $t \in T$ such that $c(t_{\alpha j}, t) \neq \emptyset$ and $t_{\beta j} \notin c(t_{\alpha j}, t)$, it follows that t is allowed, and for every sequence of firing transitions σ_j that does not contain $t_{\beta j}$ there exists h_j such that $t_{\beta j} \in \mathcal{E}(\delta\ (m, t_{\alpha j}\ \sigma_j\ h_j))$, moreover, if $m(l_E) = 1$ and $t \in \mathcal{E}(m)$, then $t \notin T_\beta$.

Lemma 1. Given M' containing a k-EM (l_E, φ). If there is a sequence σ that does not contain $t_{\beta i}$ such that $t_{\alpha i}\ \sigma\ t_{\beta i}$ is fireable from $m \in A\ (R', m_0')$, then σ does not contain $t_{\alpha j}$ for all $j \neq i$.

Proof: Upon firing $t_{\alpha i} \in \mathcal{E}(m)$, $m'(l_E) = 0$. Therefore, the resource is consumed and any $t_{\alpha j}$ with $(1 \le j \le k)$ cannot be enabled during σ. ■

Lemma 2. If M' contains a k-EM, then l_E is a safe place.

Proof: By **Definition 12**, no transition requiring l_E is enabled if $m(l_E)=0$, and release transitions are not fired if $m(l_E)=1$. Thus, $m(l_E) \le 1$. ■

Let M be a subnet of M' where the resource place l_E is removed, and where the marking is $m=\mu$ whenever $m' = (\mu, m'\ (l_E))$.

Theorem 1. If M is bounded, then M' is bounded.

Proof: Every marking $m \in A\ (R', m_0')$ implies that the sub-marking μ (without $m(l_E)$) is reachable in $A\ (R, m_0)$. Since $A\ (R, m_0)$ is bounded, μ is bounded. By **Lemma 2**, $m(l_E) \le 1$. Hence, M' is bounded. ■

Theorem 2. If M is non-blocking, then M' is non-blocking.

Proof: By induction on k. For $k=1$, since M is non-blocking, if the resource is occupied $(m(l_E)=0)$, the net M guarantees that $t_{\beta 1}$ is eventually fired, releasing it. For $k=n+1$, processes compete for l_E. Given that M is non-blocking and l_E is cyclically released, there will always be a sequence enabling any $t_{\alpha(n+1)}$. Thus, M' is non-blocking. ■

Theorem 3. If M is reversible, then M' is reversible.

Proof: Analogous to **Theorem 2**, ensuring that the resource can always be released and the base system can return to its initial state. ■

In many practical cases of modern industry, the behavior of resources is non-linear. A shared resource, such as the aforementioned Articulated Robotic Crane, frequently faces operations with choice. Upon picking up a part, the crane could deposit it at different stations (e.g., Cooling or Rework) depending on the result of an inspection. This flexibility precludes defining the exclusion with a single pair of transitions (t_α, t_β) per process. To model this reality without losing mathematical rigor, it is necessary to expand the concept by using a pair of transition sets (T_α, T_β) that encompass all possible acquisition and release actions of the resource associated with a process, thereby giving rise to the definition of generalized mutual exclusion.

Definition 13 (Generalized k-mutual exclusion). A generalized k-mutual exclusion for M' is a pair (l_E, φ) such that:

1. $l_E \in L_f$, $m_0(l_E) = 1$ and $\varphi = \{(t_{\alpha 1}, t_{\beta 1}), \ldots, (t_{\alpha k}, t_{\beta k})\}$, $k \ge 1$, is a finite set of pairs of sets that satisfy the following conditions:

(a) If $i \neq j$, $1 \le i, j \le k$, then $T_{\alpha i}, T_{\beta i} \subset T$, $T_{\alpha i} \cap T_{\beta j} = \emptyset$, $T_{\alpha i} \cap T_{\alpha j} = \emptyset$ and $T_{\beta i} \cap T_{\beta j} = \emptyset$;

(b) If $t \in T_\alpha$, $t' \in T_\beta$ and $t^* \notin T_\alpha \cup T_\beta$, then $\#\ (l_E, E(t)) = \#\ (l_E, S(t')) = 1$ and $\#\ (l_E, S(t^*)) = \#\ (l_E, E(t^*)) = 0$, where $T_\alpha = T_{\alpha 1} \cup T_{\alpha 2} \cup \ldots \cup T_{\alpha k}$ and $T_\beta = T_{\beta 1} \cup T_{\beta 2} \cup \ldots \cup T_{\beta k}$;

(c) If $t \in T_{\alpha i}$ and $t' \in T_{\beta i}$, then $c\ (t, t') \neq \emptyset$;

(d) If $t \in T_{\alpha i}$, $l \in L_v$ and $l \in c(t)$, then there exists $t' \in T_{\beta i}$ such that $t' \in c(t)$;

(e) If $t \in T_{\alpha i}$ and $c(t) \cap (L_f \cup L_v) = l_E$, then there exists $t' \in T_{\beta i}$ such that $t' \in c(t)$;

(f) If $t'' \in c\ (t, t')$, then t'' is on an o-path $c\ (t_{\alpha i})$, where $t \in T_{\alpha i}$, $t' \in T_{\beta i}$ with $1 \le i \le k$.

2. If $L_f \cup L_v$ satisfy the conditions **CL** and **S**, $t \in T_\beta$ and $t' \in T_\alpha$, then $\nexists\ c\ (t, t') = o$-path;

3. There exists a marking m_1 for a set of transitions $\{t_{\alpha 1}, \dots, t_{\alpha k}\}$ where $t_{\alpha i} \in T_{\alpha j}$ and there exists a sequence of transitions σ that does not contain any transition in $T_\alpha \setminus \{t_{\alpha 1}, \dots, t_{\alpha k}\}$ such that the marking resulting from firing σ at m_1 enables $t_{\alpha i}$ with $1 \le i \le k$.

4. For every marking m_1, if $t' \in T_{\alpha j}$ is triggered at $m \in A(R, m_1)$ then for every $t \in T$ such that $c(t', t) \neq \emptyset$ and $T_{\beta j} \cap c(t', t) = \emptyset$, it follows that t can be enabled and for every sequence of triggerable transitions σ_j that does not contain $T_{\beta j}$ there exists h_j and $t'' \in T_{\beta j}$ such that the marking reached by the triggering of $t'\sigma_j h_j$ enables t''; furthermore, if $m(l_E) = 1$ and t^* is enabled, then $t^* \notin T_\beta$.

Theorem 4. Let M be a base subnet (without the shared resource) and M' the extended net containing a generalized k-mutual exclusion (l_E, φ). If M possesses the properties of boundedness, non-blocking, or reversibility, then M' integrally preserves these properties.

Proof: It is evident from **Definition 13** that if $|t_{\alpha i}| = |t_{\beta i}| = 1$ for all i, the generalized structure collapses to the simple k-mutual exclusion, in which case the proof is given by **Theorems 1**, **2**, and **3**.

For the general case (cardinalities > 1), let us observe the marking of place l_E. By hypothesis, $m_0'(l_E)=1$. Because the input and output functions are restricted by **point 2** of **Definition 13**, any firing of a transition $t \in T_{\alpha i}$ will consume the only token in l_E $(m(l_E)=0)$. This immediately blocks the firing of any other transition in the union of all $T_{\alpha j}$ sets, ensuring that $m(l_E)$ fluctuates strictly between 0 and 1. Therefore, l_E is a safe place. Since μ (the marking of subnet M) is bounded by hypothesis, it is concluded that the global marking in M' is also bounded.

Regarding non-blocking and reversibility, suppose a process i captures the resource $(m(l_E)=0)$. Since subnet M intrinsically possesses the properties of non-blocking and cyclic behavior, process i will not stagnate in isolation. The natural evolution of M guarantees that, regardless of the choice path taken by the process, a state will inevitably be reached that enables the firing of some transition $t' \in T_{\beta i}$. Upon firing t', the resource is restored $(m(l_E)=1)$, allowing the global system to reenable the acquisition sets T_α or return to its initial marking. Consequently, the generalization of the transitions only introduces restrictions in the temporal interleaving order, without destroying the dynamic properties of M ■

5 Application

Consider an Articulated Robotic Crane at the center of a manufacturing cell, representing a shared resource. Suppose this Crane is required by three distinct processes: Assembly, Inspection, and Cooling; and the aforementioned resource is not released until the current process is completed.

Next, for modeling purposes, the construction is carried out based on **Observation 1**, the previously proposed methodology, and the interpretations of places and transitions, as follows.

1. The operations or activities are those of holding, scanning, and transporting parts by the Crane from **Input Belts A** and **B** for Assembly, from the **Precision Table** for Inspection, and from **Ovens 1**, **2**, and **3** for Cooling. In the system, the resources are the parts and the Articulated Robotic Crane;

2. The admitted activities in the system are $l_2 =$ The Crane holds, scans, transports, and assembles part **A** or **B**, $l_4 =$ The Crane holds, scans, and transports the part to be inspected, and $l_6 =$ The Crane holds and transports parts **1**, **2**, and **3**. Any of these parts is transported one before the other, where such activities do not preserve a precedence relationship;

3. For a part **A** or **B**, the operation of holding, scanning, transporting, and assembling in **Modules 1** or **2** is given in **Figure 1**. The transitions represent the following: $t^1_{\alpha 1} =$ Start of activity l_2 from **Input Belt A**, $t^2_{\alpha 1} =$ Start of activity l_2 from **Input Belt B**, $t^1_{\beta 1} =$ End of activity l_2 to **Module 1**, $t^2_{\beta 1} =$ End of activity l_2 to **Module 2**, and $t^3_{\beta 1} =$ End of activity l_2 to the Defect **Rejection Bin**.

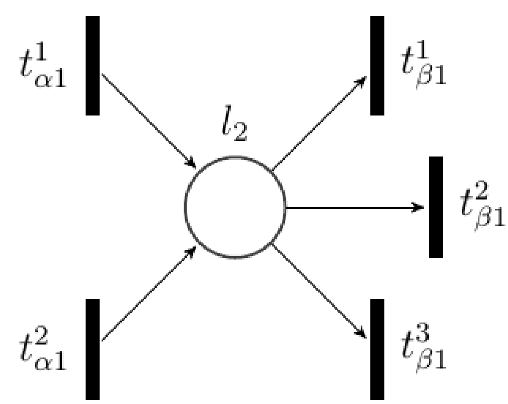

Figure 1: A **PN** to model operation l_2.

For the **Precision** part, the operation of holding, scanning, transporting, and inspecting to **Approve** or **Rework** is given in **Figure 2**. The transitions represent the following: $t^1_{\alpha2}=$ Start of activity l_4 from the **Precision Table**, $t^1_{\beta2}=$ End of activity l_4 to the **Approved Belt**, $t^2_{\beta2}=$ End of activity l_4 to the **Rework Table**.

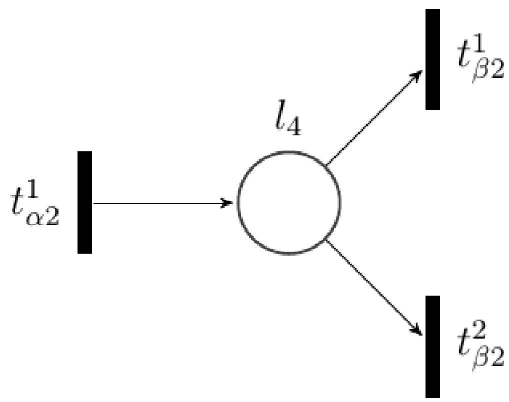

Figure 2: A **PN** to model operation l_4.

For the **Hot** part, the operation of holding and transporting for **Cooling** is given in **Figure 3**. The transitions represent the following: $t^1_{\alpha3}=$ Start of activity l_6 from **Oven 1**, $t^2_{\alpha3}=$ Start of activity l_6 from **Oven 2**, $t^3_{\alpha3}=$ Start of activity l_6 from **Oven 3**, $t^1_{\beta3}=$ End of activity l_6 to the **Drying Belt**.

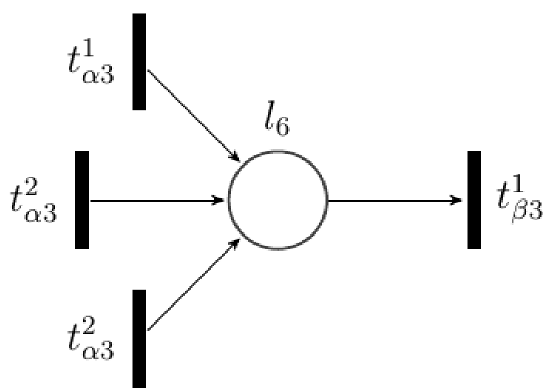

Figure 3: A **PN** to model operation l_6.

The marking of l_2, l_4 and l_6 establishes the occurrence of the operation in question;

4. To carry out operations l_2, l_4 and l_6, the availability of the relevant part and the Articulated Robotic Crane is required. Let $l_1 =$ A part **A** or **B** is waiting to be assembled, $l_3 =$ A **Precision** part is waiting to be inspected, $l_5 =$ A **Hot** part is waiting to be cooled, and $l_E =$ The Articulated Robotic Crane is available to carry out its task. Consequently, places l_1, l_3 and l_5 are tied to their respective starting transitions, and l_E to all of these without exception; then, l_1, l_3, l_5 and l_E become available upon the completion of any activity. Thus, for **Processes 1**, **2**, and **3**, and taking **i=1,3,5**, two directed arcs are constructed from each completion transition to l_E and its respective place l_i. This is shown in **Figures 4**, **5**, and **6**. If place l_i is marked with a token, then the part is available; now, if place l_i is not marked, then the respective part is not available. On the other hand, if place l_E is marked, then the Crane is available; otherwise, it is not. Furthermore, if l_i and l_E are marked, then the processes are executable, and in this case, the preconditions for their occurrence are met;

5. Let us consider the initial marking $m_0 = (1,0,1,0,1,0,1)$, which simultaneously determines the availability of the Crane and the parts to be processed; note that $l_E = l_7$ for our **7**-tuples of marking.

Observation 2. The previously described manufacturing system possesses a generalized **3**-mutual exclusion, where $(l_E, \varphi) = (l_E, \{(T_{\alpha1}, T_{\beta1}), (T_{\alpha2}, T_{\beta2}), (T_{\alpha3}, T_{\beta3})\})$, with $T_{\alpha1} = \{t^1_{\alpha1}, t^2_{\alpha1}\}$, $T_{\beta1} = \{t^1_{\beta1}, t^2_{\beta1}, t^3_{\beta1}\}$, $T_{\alpha2} = \{t^1_{\alpha2}\}$, $T_{\beta2} = \{t^1_{\beta2}, t^2_{\beta2}\}$, $T_{\alpha3} = \{t^1_{\alpha3}, t^2_{\alpha3}, t^3_{\alpha3}\}$ y $T_{\beta3} = \{t^1_{\beta3}\}$. In addition, **L** is partitioned as $L = L_o \cup L_f \cup L_v$, where $L_o = \{l_2, l_4, l_6\}$, $L_f = \{l_E\}$ and $L_v = \{l_1, l_3, l_5\}$. To conclude, the complete **PN** of the system is obtained by coupling the **3** nets to the fixed place l_E from **Figures 4**, **5**, and **6**.

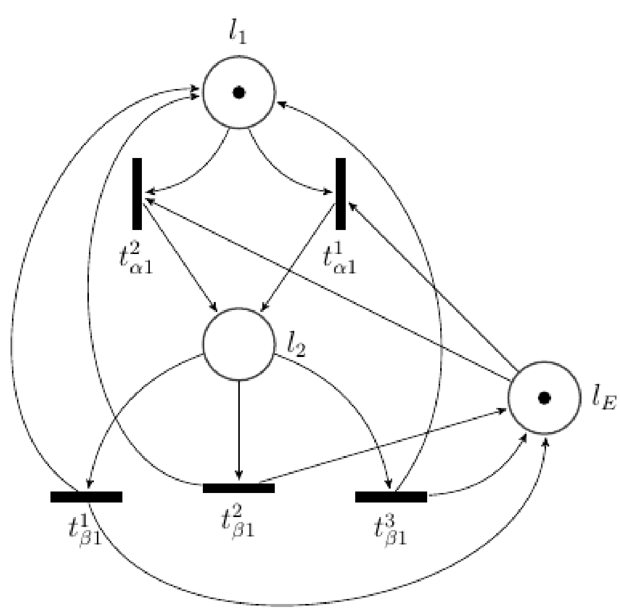

Figure 4: A **PN** to model the assembly process.

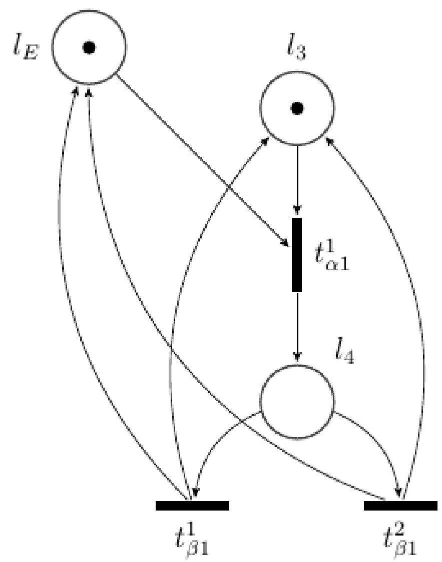

Figure 5: A **PN** to model the inspection process.

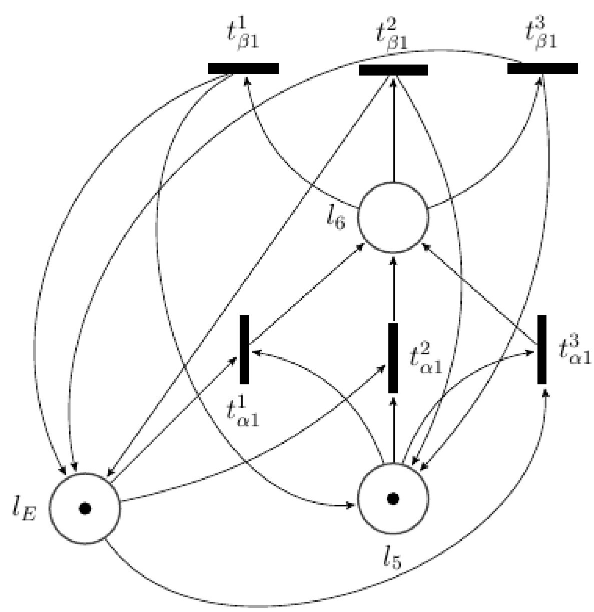

Figure 6: A **PN** to model the cooling process.

6 Conclusions

The mathematical development of generalized mutual exclusion, transcending the limitations of simple exclusion, responds directly to the demands of modern smart manufacturing.

As demonstrated in **Section 5**, the application of this theory to an Articulated Robotic Crane illustrates how complex and integrated processes—such as Assembly, Inspection, and Cooling—can be successfully modeled, even when shared resources are required to make real-time decisions and select alternative routes.

The formulated preservation theorem (**Theorem 4**) provides an invaluable guarantee: formally certifying that the superposition of this generalized mutual exclusion onto a qualitatively stable process will preserve, without exception, the boundedness, deadlock-freeness, and reversibility of the global system. This synergy between Petri Net algebra and automation allows engineers to directly synthesize—bypassing trial-and-error approaches—supervisory control code that is strictly free from deadlocks and overflows.

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Rosales Moret, José René. *Graduate in Mathematics. Master of Science in Mathematics. Active Full Professor. Faculty of Sciences, ULA. Department of Mathematics.*
https://orcid.org/0009-0006-8265-8181

Mata-Diaz, Guelvis E. *Graduate in Mathematics. Master of Science in Mathematics. PhD in Applied Sciences. Active Full Professor. Faculty of Sciences, ULA. Department of Mathematics. Research area: analysis and control in discrete-event dynamic systems. E-mail: gematad2017@gmail.com*
https://orcid.org/0000-0001-7147-1422

Ferrer-Guillén, María Dolores. *Master's degree in Computing. Active Full Professor. Department of Calculation, Faculty of Engineering, ULA. Research area: analysis and control in dynamic systems. E-mail:* mariadfg@gmail.com
https://orcid.org/0009-0002-8162-233X

Mantilla Morales, Gisella. *Engineer in Electronics and Telecommunications, Master's degree in Mathematics; full-time professor at the Technical University of Manabí. E-mail: gisella.mantilla@utm.edu.ec*
https://orcid.org/0000-0002-0826-7741

Coello Vera, Adalberto Antonio. *Engineer in Agricultural Business Administration, Master's degree in Mathematics; professor at the Technical University of the State of Quevedo. E-mail: adalbertocoellov@hotmail.com*
https://orcid.org/0000-0001-5985-3532

Perez-Pirela, Maribel. *Chemical engineer. Active Full Professor. Department of Mathematics and Statistics, Faculty of Basic Sciences, Technical University of Manabí. Research area: Mathematics and Statistics. E-mail: maribel.perez@utm.edu.ec*
https://orcid.org/0000-0002-9687-5471